Lecture Notes · Unit 7 · Lecture 7 AP CED 7.8

Exponential

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INTRO: PROPORTIONAL TO ITS SIZE

Exponential models (AP 7.8)

Let's get that 5

IDEA 1: SOLVE DY/DT = KY

Key ideas

Grows in proportion to itself → what function?

1. Solve it

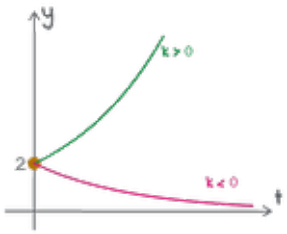
$\frac{dy}{dt} = ky$

$\frac{1}{y}\,dy = k\,dt \rightarrow \ln|y| = kt + C$

Raise e → $y = Ae^{kt}$; $t = 0 \rightarrow y = A$, the start

$y = y_0 e^{kt}$, y_0 at $t = 0$

$y_0 = 2$



$k > 0$ → exponential growth $k < 0$ → decay

$y_0 e^{kt}$ only for $\frac{dy}{dt} = ky$ vs. $k(T - 70)$

IDEA 2: K FROM TWO VALUES

Key ideas

2. Solve for k

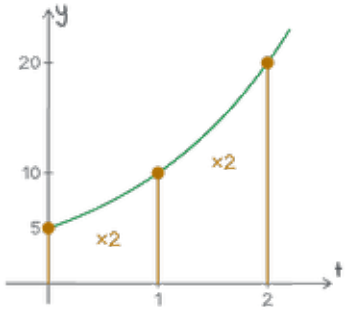
$y = y_0 e^{kt}$

$y(0) = 5$, $y(2) = 20 \rightarrow 20 = 5e^{2k} \rightarrow$ k from two values

$e^{2k} = 4 \rightarrow 2k = \ln 4 \rightarrow k = \ln 2$ (no decimals)

$e^{(\ln 2)t} = 2^t \rightarrow y = 5 \cdot 2^t$

$t = 0 \rightarrow 5$ $t = 1 \rightarrow 10$ $t = 2 \rightarrow 20$



Every 1 → ×2 → doubling time 1

NO STARTING AMOUNT?

Key ideas

No starting amount?

$y = y_0 e^{kt}$, y_0 ?

$y(1) = 6$, $y(3) = 24 \rightarrow$ start unknown

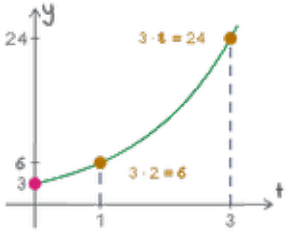
$\frac{y_0 e^{3k}}{y_0 e^{k}} = \frac{24}{6}$

$e^{2k} = 4 \rightarrow k = \ln 2$

$y(1) \rightarrow y_0 e^{k} = 6$, $e^{k} = 2$

$2y_0 = 6 \rightarrow y_0 = 3$, the start

$y = 3 \cdot 2^t \rightarrow y(1) = 6$ ✓, $y(3) = 24$ ✓



IDEA 3: THE RATE FROM THE AMOUNT

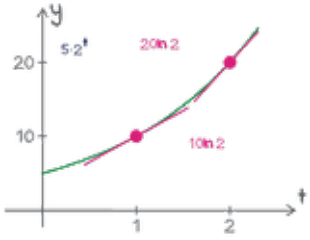
Key ideas

3. The rate from the amount

$\frac{dy}{dt} = (\ln 2)y$

$\frac{dy}{dt} = ky = (\ln 2)y$

$y = 10 \to 10\ln 2;\ y = 20 \to 20\ln 2$



or: $y = 5 \cdot 2^t \to y' = 5 \cdot 2^t \ln 2 \to t = 1 \to 10\ln 2$

Same answer → the equation is shorter

$y = 10 \to 10\ln 2$ per unit time → growing

AP → "rate proportional to its size"

PROBLEM 1 OF 2

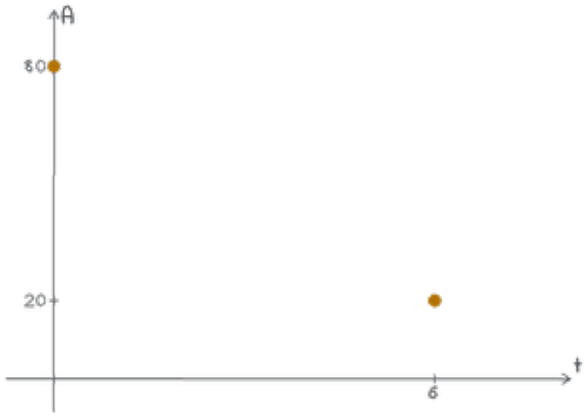
Decay of a drug

A patient has 80 mg of a drug. The amount A decreases at a rate proportional to A, and after 6 hours, 20 mg remain.

(a) Write $A(t)$ and find k.

(b) When do 10 mg remain?

(c) Find $\frac{dA}{dt}$ when $A = 40$. Explain its meaning in context.



Solution

Exponential decay: k from the second value, then halvings, then the rate

Plan: $A = 80e^{kt} \to k \to$ halvings → rate

(a)

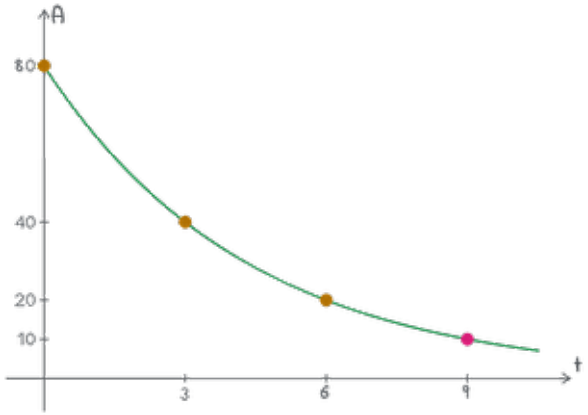
$A = 80e^{kt} \to 20 = 80e^{6k}$

$e^{6k} = \frac{1}{4} \to 6k = -\ln 4 = -2\ln 2 \to k = -\frac{\ln 2}{3}$

$A = 80e^{-\frac{\ln 2}{3}t}$ (b) → half-life?

Solution

Exponential decay: k from the second value, then halvings, then the rate



(b) time → (c) rate at one amount

(b)

$e^{3k} = e^{-\ln 2} = \frac{1}{2} \to$ half every 3 h

$80 \to 40 \to 20 \to 10 \to t = 9$

(c)

$\frac{dA}{dt} = kA = -\frac{\ln 2}{3} \cdot 40 = -\frac{40\ln 2}{3}$

$A = 40 \to$ decreasing $\frac{40\ln 2}{3}$ mg per hour

Two values, k, the rate

Careful

X Common mistake

$\frac{60}{6} = 10$

Always −10 mg per hour?

✓ Do this

$\times\frac{1}{2}$ every 3 h

80 → 40 → 20 mg

PROBLEM 2 OF 2

Three times

An amount grows at a rate proportional to itself and doubles every 5 years. When is it 3 times its starting size?

(A) 7.5

(B) $\frac{5\ln 3}{\ln 2}$

(C) 15

(D) $\frac{5\ln 2}{\ln 3}$

Solution

Write it with the doubling time, set it to three times, take logs

Plan: $y = y_0 \cdot 2^{\frac{t}{5}} \rightarrow$ set $= 3y_0$

$\frac{t}{5}$ = doublings (5 → 1) → $y = y_0 \cdot 2^{\frac{t}{5}}$

$2^{\frac{t}{5}} = 3 \rightarrow \frac{t}{5}\ln 2 = \ln 3$

$t = \frac{5\ln 3}{\ln 2}$ $\quad 5 < t < 10$ $\quad t \approx 7.9$

So (B).

Doublings: one more in the workbook

WRAP-UP

An exponential

$\frac{dy}{dt} = ky \rightarrow y = y_0 e^{kt}$

Two values → k (leave the ln)

Rate → k times the amount

Workbook → video → comments

Next → Unit 8

Next: the workbook for this topic (link in the video description). Stuck? Ask in the comments.