7.8 / Exponential Models with Differential Equations

Watch the lesson ➤

$\frac{dy}{dt} = ky$ gives $y = y_0e^{kt}$; k from two values; the rate from the amount

Learning goal

Use the model $\frac{dy}{dt} = ky$ and its solution $y = y_0e^{kt}$: find k from two values, work with doubling times and half-lives, and find the rate from the amount.

Key ideas

01 Rate proportional to the amount itself: $\frac{dy}{dt} = ky$, so $y = y_0e^{kt}$. y_0 is the amount at $t = 0$.

02 $k > 0$: **exponential growth**. $k < 0$: exponential decay toward 0.

03 Find k from two values: e^{kT} = the ratio. Leave k as a natural log (no decimals). Two values with no start: divide, y_0 cancels.

04 Equal times multiply by the same factor: **doubling time**, **half-life**. $e^{(\ln 2)t} = 2^t$.

05 The rate right then: $\frac{dy}{dt} = ky$ with the amount. y_0e^{kt} is only for $\frac{dy}{dt} = ky$, not for $k(T - 70)$.

Remember

exponential growth / decay
$y = y_0e^{kt}$ with $k > 0$ / $k < 0$.

half-life
The time for the amount to be cut in half.

doubling time
The time for the amount to double.

Class notes

7.8 / Write it clearly

What earns the point on the AP Exam, and the mistakes that lose it.

Writing checklist

Notation to use and to avoid

WRITE	AVOID
Write k exactly: $k = -\frac{\ln 2}{3}$.	Do not round k to a decimal before the last step.
Give the rate with units: "decreasing at $\frac{40 \ln 2}{3}$ mg per hour."	Do not use $y_0 e^{kt}$ when the rate is proportional to $T - 70$.

Common mistakes

The mistake code is used in the answers and in My Mistakes

01 Not a straight line

Code DE12

✗ Doubles every 5 years, so it is 3 times its size after 7.5 years.

✓ $2^{t/5} = 3$, so $t = \frac{5 \ln 3}{\ln 2} \approx 7.9$ years.

02 Only for a rate times the amount

Code DE12

✗ $\frac{dT}{dt} = -\frac{1}{2}(T - 70)$, $T(0) = 150$: $T = 150e^{-t/2}$.

✓ The rate is proportional to $T - 70$: $T = 70 + 80e^{-t/2}$ (Lesson 7.7).

Real AP questions on this topic Free on AP Central. Try them after the practice.

Exponential models appear in multiple-choice questions; no released free-response part on 7.8 alone is listed here.

7.8 / Guided examples

Watch, pause, and write the reasoning. Full solutions are in the video.

Problem types

Read the problem, name the type, then use its method

Type	Method	Problems
$y = y_0 e^{kt}$ and k	Only for a rate k times the amount; put in a second value; e^{kT} = ratio *o* k with a natural log.	E01 · Practice 1, 2, 5, 7
The rate from the amount; growth by factors	$\frac{dy}{dt} = ky$ at that amount; equal times multiply by the same factor (doubling, halving).	E02 · Practice 3, 4, 6

Example 01

Video · E01

A patient has 80 mg of a drug. The amount A decreases at a rate proportional to A, and after 6 hours, 20 mg remain.
(a) Write $A(t)$ and find k.
(b) When do 10 mg remain?
(c) Find $\frac{dA}{dt}$ when $A = 40$. What does it mean?

> Two values give k. The rate comes from the amount: $\frac{dA}{dt} = kA$.

Example 02

Video · E02

An amount grows at a rate proportional to itself and doubles every 5 years. When is it 3 times its starting size?

A. 7.5
B. $\frac{5 \ln 3}{\ln 2}$
C. 15
D. $\frac{5 \ln 2}{\ln 3}$

> Doubling every 5 years: $y = y_0 \cdot 2^{t/5}$. Set it equal to $3y_0$.

VIDEO COMPLETE Check your example answers at the end of this booklet.

7.8 / Example answers

Exponential Models with Differential Equations

E01 (a) $A(t) = 80e^{kt}$, $k = -\frac{\ln 2}{3}$; (b) $t = 9$ hours; (c) $-\frac{40 \ln 2}{3}$: when $A = 40$ mg, the amount is decreasing at $\frac{40 \ln 2}{3}$ mg per hour

E02 (B)