Lecture Notes · Unit 7 · Lecture 4 AP CED 7.4

Reasoning

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INTRO: RISE? BEND? OVER OR UNDER?

Reasoning with Slope Fields AP Calculus 7.4

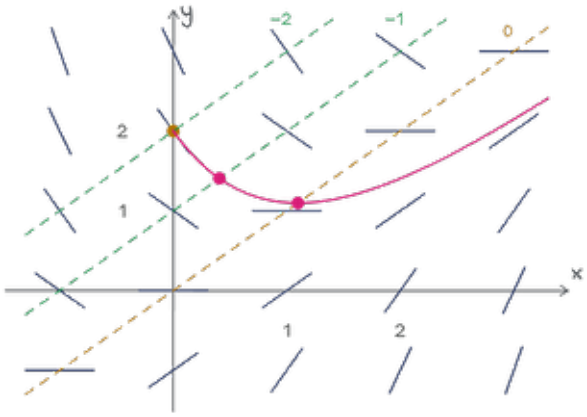
$\frac{dy}{dx} = \cos y \rightarrow$ no solving

→ rise? bend? over or under?

AP 5

IDEA 1: FOLLOW THE SEGMENTS

Follow the segments



$\frac{dy}{dx} = x - y \rightarrow$ start at $(0,2) \rightarrow$ ride the flow

$(0,2)$ on $y = x + 2 \rightarrow -2 \rightarrow$ steep down

on $y = x + 1 \rightarrow -1$, gentler

on $y = x \rightarrow 0 \rightarrow$ the bottom

Past the bottom → $x > y \rightarrow + \rightarrow$ up

CURVES NEVER CROSS

From $(0,-1) \rightarrow$ slope 1 all the way $\rightarrow y = x - 1$

$\frac{dy}{dx} = 1, x - (x-1) = 1 \rightarrow \checkmark$

one slope per point → curves don't cross

Alongside the segments → solution curve

curve slope = segment slope at each point

AP → sketch the curve through a given point

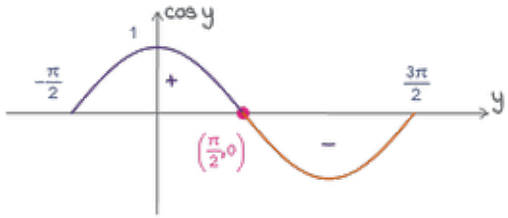
IDEA 2: RISING, FALLING, CONSTANT

The sign of the slope

$\frac{dy}{dx} = \cos y \rightarrow$ no field, no solving

The sign of the slope

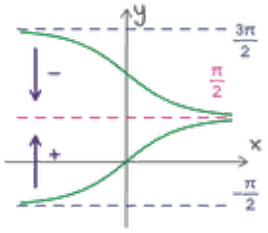
Slope meter: cos y against y



y in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow + \rightarrow$ rising

$\left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \rightarrow - \rightarrow$ falling

$y = \frac{\pi}{2} \rightarrow \frac{dy}{dx} = 0$ for every x



Flat line → constant solution

IDEA 3: CONCAVITY, OVER OR UNDER

Bend → over or under

The second derivative → over or under

$\frac{dy}{dx} = \cos y$

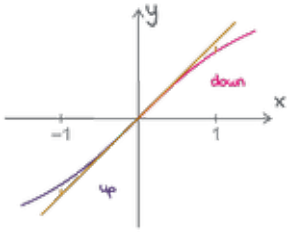
origin → $y(0) = 0$

$\frac{d}{dx}\cos y = -\sin y \cdot \frac{dy}{dx}$

$\frac{d^2y}{dx^2} = -\sin y \cos y$

$0 < y < \frac{\pi}{2} \rightarrow -(+)(+) < 0 \rightarrow$ concave down

$-\frac{\pi}{2} < y < 0 \rightarrow -(-)(+) > 0 \rightarrow$ concave up



tangent above → over

tangent below → under

Over or under

$\frac{dy}{dx} = \cos y$

$(0,0) \rightarrow \cos 0 = 1 \rightarrow y = x$

No formula → use the tangent → read at ±0.2

$y(0.2) \approx 0.2$, $y(-0.2) \approx -0.2$

$0 < x \leq 0.2 \rightarrow$ climbs from $0 \rightarrow 0 < y < \frac{\pi}{2}$

$y'' < 0 \rightarrow$ concave down → tangent above 0.2 over

$-0.2 \leq x < 0 \rightarrow$ climbs to $0 \rightarrow -\frac{\pi}{2} < y < 0$

$y'' > 0 \rightarrow$ concave up line below −0.2 under

Concavity whole interval, not one point

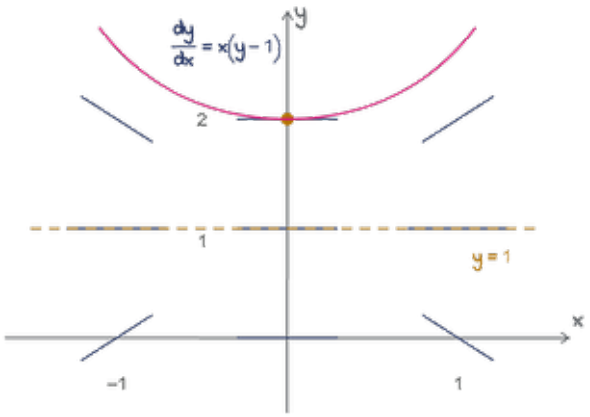
2023 AB 3(c) $\frac{d^2y}{dx^2}$ at one point → whole interval

PROBLEM 1 OF 4

The flat-line fence

The slope field for $\frac{dy}{dx} = x(y-1)$ is shown at nine points.

Sketch the solution curve through $(0, 2)$.



Solution

Start at the point, follow the segments, check the flat row

Plan: start at $(0,2)$ → follow

$(0,2) \rightarrow 0 \cdot (2-1) = 0 \rightarrow$ flat

Right + left − → up both ways

A valley lowest at $(0,2)$

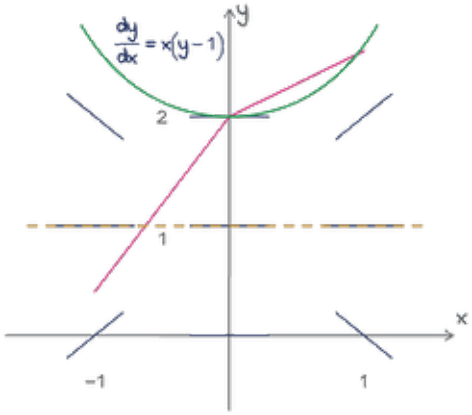
$y = 1 \rightarrow x \cdot 0 = 0 \rightarrow (1)' = 0 \rightarrow$ a solution

Flat line that solves it → constant solution

Above $y = 1 \rightarrow$ stays above

Workbook → start, follow, flat rows

Careful → ride, don't connect



✗ Common mistake

Corners ✗

y = 1 crossed ✗

✓ Do this

Smooth ✓

y > 1 ✓

PROBLEM 2 OF 4

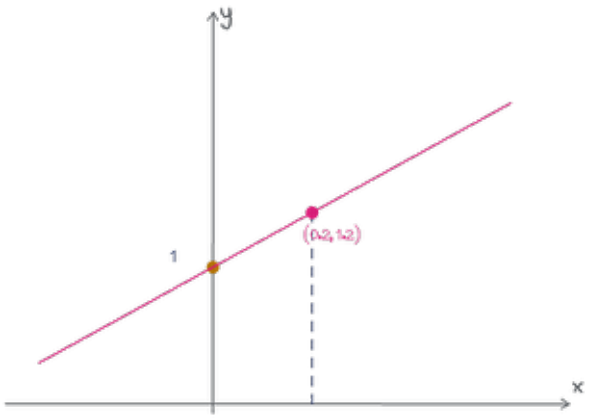
Tangent, concavity, extremum

Let $y = f(x)$ be the solution to $\frac{dy}{dx} = y^2 - x$ with $f(0) = 1$.

(a) Use the tangent line at $x = 0$ to approximate $f(0.2)$.

(b) Find y'' in terms of x, y. Is it concave up or down at $(0, 1)$?

(c) Another solution passes through $(4, 2)$. Is it a relative max, a relative min, or neither? Justify your answer.



Solution Plug in the point → tangent line → read it

Plan: point → tangent; again → y''

(a)

$(0, 1) \to 1^2 - 0 = 1 \to y = 1 + x$

$f(0.2) \approx 1 + 0.2 = 1.2$

Workbook → point in first

Solution

Differentiate the first derivative again, plug it back in, then test each point

Plan: y'', then each point

(b)

$$\frac{d^2y}{dx^2} = 2y\frac{dy}{dx} - 1$$

$$= 2y\left(y^2 - x\right) - 1$$

$(0, 1) \to 2 \cdot 1 \cdot 1 - 1 = 1 > 0 \to$ concave up

(c)

$(4, 2) \to \frac{dy}{dx} = 4 - 4 = 0 \to$ flat, not enough

$y'' = 2 \cdot 2 \cdot 0 - 1 = -1 < 0$

$\frac{dy}{dx} = 0, \frac{d^2y}{dx^2} < 0 \to$ relative maximum

More in your workbook

Careful → flat is not enough

X Common mistake

$y' = 0 \rightarrow$ max?

Slope field alone?

✓ Do this

$y' = 0, y'' = -1 < 0$

→ relative maximum

PROBLEM 3 OF 4

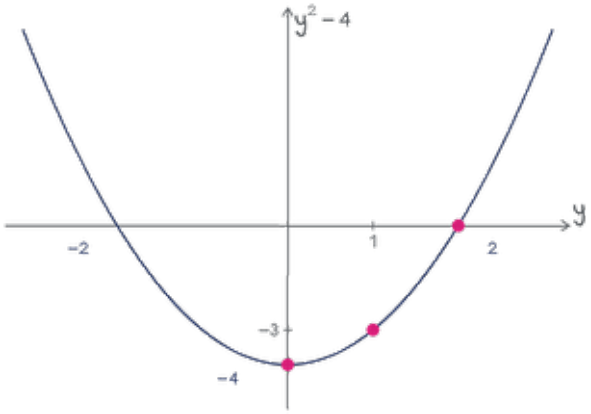
Three statements

For $\frac{dy}{dx} = (y^2 - 4)e^x$, which of the following are true?

I. $y = 2$ is a solution.

II. The solution with $y(0) = 0$ is decreasing at $x = 0$.

III. The solution through $(0, 1)$ has a relative min at $x = 0$.



Solution

Read the sign off the meter; e^x is always positive

Plan: graph $y^2 - 4$; $e^x > 0$ always

I. $y = 2 \rightarrow \frac{dy}{dx} = 0$ $(2)' = 0 \rightarrow$ constant solution ✓

II. $y = 0 \rightarrow -4 \cdot e^0 < 0 \rightarrow$ decreasing ✓

III. $y = 1 \rightarrow -3 \neq 0 \rightarrow$ not a critical point X

So I and II only.

Read each sign off a graph

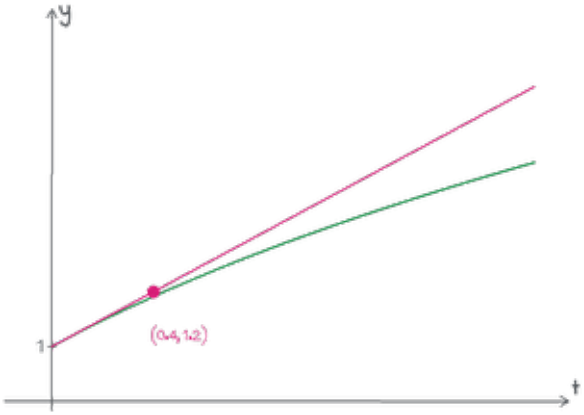
PROBLEM 4 OF 4

Over or under?

Let y satisfy $\frac{dy}{dt} = \frac{1}{1+y^2}$ with $y(0) = 1$, for $t \geq 0$.

(a) Use the tangent line at $t = 0$ to approximate $y(0.4)$

(b) Is it an overestimate or an underestimate? Explain your reasoning.



Solution Tangent line, then the second derivative

Plan: tangent → sign of y'' on interval

(a)

$(0, 1)$ $\frac{1}{1+1} = \frac{1}{2} \rightarrow 1 + 0.4 \cdot \frac{1}{2} = 1.2$

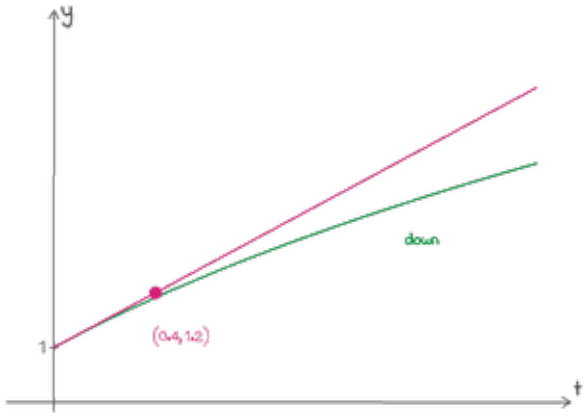
(b)

$$\frac{d^2y}{dt^2} = \frac{-2y}{(1+y^2)^2} \cdot \frac{dy}{dt}$$

$$= \frac{-2y}{\left(1+y^2\right)^3}$$

Solution

The sign of the second derivative on the whole interval



Plan: sign of y'' on $0 \le t \le 0.4$

$\frac{dy}{dt} > 0 \rightarrow$ y goes up from 1

$t \ge 0 \rightarrow y \ge 1 > 0$

$y'' < 0$ on $0 \le t \le 0.4 \rightarrow$ concave down

Tangent above → 1.2 → overestimate

Workbook
3 pieces

WRAP-UP

Rise? Bend? over or under?

Follow the segments → curves don't cross

Sign of $\frac{dy}{dx}$ → rising, falling; $\frac{dy}{dx} = 0$ for every x → constant

y'' → chain rule, plug $\frac{dy}{dx}$ back in

over or under → y'' on the whole interval

Workbook → video → comments

Next: 7.5 BC only → 7.6 separation of variables

Next: the workbook for this topic (link in the video description). Stuck? Ask in the comments.