7.4 / Reasoning Using Slope Fields and Differential Equations

Watch the lesson ➤

Solution curves; the equation alone gives the sign, concavity, and over or under

Learning goal

Sketch a solution curve on a slope field and, without solving, use the differential equation for tangent lines, extrema, concavity, and whether an approximation is over or under.

Key ideas

01 A **solution curve** starts at its point and follows the segments. It never crosses a flat row that is a constant solution.

02 A flat line $y = c$ that makes both sides 0 is a **constant solution**.

03 Point in first: put the point into $\frac{dy}{dx}$ for the slope of the tangent line. No solving.

04 For $\frac{d^2y}{dx^2}$, differentiate with y a function of x, then put $\frac{dy}{dx}$ back in.

05 Over or under: the sign of $\frac{d^2y}{dx^2}$ on the **whole interval** from the point to the estimate. $\frac{dy}{dx} = 0$ alone is not an extremum.

Remember

solution curve
The graph of one solution; it follows the slope field.

constant solution
A flat line $y = c$ that solves the equation.

overestimate
The tangent line is above the curve: concave down on the interval.

Class notes

7.4 / Write it clearly

What earns the point on the AP Exam, and the mistakes that lose it.

Writing checklist

Notation to use and to avoid

WRITE	AVOID
Name both pieces: "$\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$ at $(4, 2)$, so a relative maximum."	Do not decide over or under from the sign at one point only.
For over or under, say where: "$\frac{d^2y}{dt^2} < 0$ for $0 \le t \le 0.4$."	Do not draw a solution curve with corners or across the segments.

Common mistakes

The mistake code is used in the answers and in My Mistakes

01 The whole interval, not one point

Code DE6

✗ $\frac{d^2y}{dx^2} < 0$ at $x = 0$, so the estimate of $y(0.4)$ is an overestimate.

✓ Show $\frac{d^2y}{dx^2} < 0$ for every x from 0 to 0.4; then it is an overestimate.

02 Put dy/dx back in

Code DE7

✗ $\frac{dy}{dx} = y^2 - x$, so $\frac{d^2y}{dx^2} = 2y - 1$.

✓ y depends on x: $\frac{d^2y}{dx^2} = 2y\frac{dy}{dx} - 1 = 2y(y^2 - x) - 1$.

03 Flat is not enough

Code DE14

✗ $\frac{dy}{dx} = 0$ at $(4, 2)$, so the solution has a maximum there.

✓ Also $\frac{d^2y}{dx^2} = -1 < 0$ there (or a sign change): then a relative maximum.

Real AP questions on this topic Free on AP Central. Try them after the practice.

- 2023 AB FRQ 3(a-c) – a solution curve on a slope field, a tangent-line estimate, concavity over the interval

7.4 / Guided examples

Watch, pause, and write the reasoning. Full solutions are in the video.

Problem types

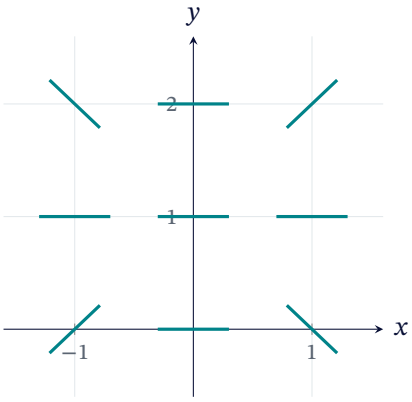
Read the problem, name the type, then use its method

Type	Method	Problems
A point, a tangent line, over or under	Put the point into $\frac{dy}{dx}$; tangent line; $\frac{d^2y}{dx^2}$ with $\frac{dy}{dx}$ put back; its sign on the whole interval.	E02, E04 · Practice 3, 5, 7
Signs, constant solutions, extrema	Flat line in → constant solution; $\frac{dy}{dx} = 0$ plus a sign change or the sign of $\frac{d^2y}{dx^2}$.	E03 · Practice 4, 6, 8
A solution curve on a slope field	Start at the point, follow the segments, never cross a flat row that is a solution.	E01 · Practice 1, 2

Example 01

Video · E01

The slope field of $\frac{dy}{dx} = x(y-1)$ at nine points.
Sketch the solution curve through $(0, 2)$.



Start at the point; follow the segments; look for a flat row.

Example 02

Video · E02

Let $y = f(x)$ be the solution to $\frac{dy}{dx} = y^2 - x$ with $f(0) = 1$.
(a) Tangent line at $x = 0$; approximate $f(0.2)$.
(b) y'' in x, y; concave up or down at $(0, 1)$?
(c) Another solution through $(4, 2)$: max, min, or neither? Justify.

> Point in first. For y'', differentiate y^2 as $2y\frac{dy}{dx}$, then put $\frac{dy}{dx}$ back in.

Example 03

Video · E03

$\frac{dy}{dx} = (y^2 - 4)e^x$. Which are true?
I. $y = 2$ is a solution.
II. $y(0) = 0 \rightarrow$ decreasing at $x = 0$.
III. Through $(0, 1) \rightarrow$ relative min at $x = 0$.

> Put each line or point into the equation and read the sign.

Example 04

Video · E04

$\frac{dy}{dt} = \frac{1}{1+y^2}$ with $y(0) = 1$, for $t \geq 0$.
(a) Tangent line at $t = 0 \rightarrow$ approximate $y(0.4)$
(b) Overestimate or underestimate? Explain.

Over or under: the sign of $\frac{d^2y}{dt^2}$ on the whole interval, from $t = 0$ to $t = 0.4$.

VIDEO COMPLETE Check your example answers at the end of this booklet.

7.4 / Example answers

Reasoning Using Slope Fields and Differential Equations

E01 flat at $(0, 2)$, a valley above the flat row $y = 1$

E02 (a) $y = 1 + x$; $f(0.2) \approx 1.2$; (b) $\frac{d^2y}{dx^2} = 2y(y^2 - x) - 1$; $1 > 0$ at $(0, 1)$: concave up; (c) relative maximum: $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = -1 < 0$ at $(4, 2)$

E03 I and II only

E04 (a) $y = 1 + \frac{t}{2}$; $y(0.4) \approx 1.2$; (b) overestimate: $\frac{d^2y}{dt^2} = -\frac{2y}{(1+y^2)^3} < 0$ on $0 \le t \le 0.4$ $(y \ge 1)$