6.10 / Integrating Using Long Division and Completing the Square

Watch the lesson ➤

Top degree at least the bottom's: divide first; no real zeros: complete the square, then arctan

Learning goal

Rewrite before integrating: divide when the top degree is at least the bottom degree, and complete the square to reach $\frac{1}{a}\arctan\frac{u}{a}$.

Key ideas

01 No basic rule fits, and the top is not the derivative of the bottom: not ln right away. Top degree = bottom degree: divide. $x+3=(x+1)+2$, so $\frac{x+3}{x+1}=1+\frac{2}{x+1}$ and $\int = x+2\ln|x+1|+C$.

02 Complete the square: $x^2+bx+c=\left(x+\frac{b}{2}\right)^2+c-\frac{b^2}{4}$. $x^2+6x+10=(x+3)^2+1$, $u=x+3$: $\arctan(x+3)+C$.

03 When $a\neq 1$: $\int\frac{1}{x^2+4}\,dx=\frac{1}{4}\int\frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx$; $u=\frac{x}{2}$, $dx=2\,du$: $\frac{1}{2}\arctan\frac{x}{2}+C$.

04 In general ($a>0$): $\int\frac{1}{u^2+a^2}\,du=\frac{1}{a}\arctan\frac{u}{a}+C$.

Remember

long division

top = bottom × quotient + remainder, so $\frac{\text{top}}{\text{bottom}}=\text{quotient}+\frac{\text{remainder}}{\text{bottom}}$.

completing the square

$x^2+bx+c=\left(x+\frac{b}{2}\right)^2+c-\frac{b^2}{4}$.

Class notes

6.10 / Write it clearly

What earns the point on the AP Exam, and the mistakes that lose it.

Common mistakes

The mistake code is used in the answers and in My Mistakes

01 ln right away, or the remainder dropped

✗ $\int \frac{x^2+3}{x+1}\,dx$: $u = x+1$, so $\ln|x+1|$; or divide and drop the remainder.

✓ Top degree ≥ bottom: divide. Quotient $+\frac{\text{remainder}}{\text{bottom}}$: $x - 1 + \frac{4}{x+1}$.

6.10 / Guided examples

Watch, pause, and write the reasoning. Full solutions are in the video.

Problem types

Read the problem, name the type, then use its method

Type	Method	Problems
Divide first	Top degree >= bottom degree: write top = bottom x quotient + remainder, then integrate piece by piece.	E01 · Practice 1, 3, 6
Complete the square	No real zeros: $\left(x+\frac{b}{2}\right)^2+a^2$, $u=x+\frac{b}{2}$, then $\frac{1}{a}\arctan\frac{u}{a}$.	E02 · Practice 2, 4, 5

Example 01

Video · E01

Find $\int \frac{x^2+3}{x+1}\,dx$.

> Top degree ≥ bottom degree: divide first.

Example 02

Video · E02

$\int_2^4 \frac{1}{x^2-4x+8}\,dx$ is

A. $\frac{\pi}{4}$

B. $\frac{\pi}{8}$

C. $\frac{\pi}{16}$

D. $\frac{1}{2}\ln 2$

> The bottom has no real zeros: complete the square.

VIDEO COMPLETE Check your example answers at the end of this booklet.

6.10 / Example answers

Integrating Using Long Division and Completing the Square

E01 $x^2 + 3 = (x+1)(x-1) + 4$

$$\frac{x^2+3}{x+1} = x - 1 + \frac{4}{x+1}$$

$$\int \left(x - 1 + \frac{4}{x+1}\right) dx = \frac{x^2}{2} - x + 4\ln|x+1| + C$$

E02 (B) $x^2 - 4x + 8 = (x-2)^2 + 4$

$u = x - 2$: ends 0 to 2, and $a = 2$.

$$\int_0^2 \frac{du}{u^2+4} = \frac{1}{2}\left[\arctan\frac{u}{2}\right]_0^2 = \frac{1}{2}\cdot\frac{\pi}{4} = \frac{\pi}{8}$$

Answer B.