Lecture Notes · Unit 6 · Lecture 9 AP CED 6.9

Integrating using substitution

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INTRO: CAN WE UNDO THE CHAIN RULE?

u-Substitution AP Calculus 6.9

$\int 2x\cos(x^2)\,dx$

No basic rule fits

Built by the chain rule → undo it?

Pick u → change the ends

INSIDE A FUNCTION: PICK U

Inside a function → u

$\int \cos(x^2)\,2x\,dx$

Look first → inside function

Inside: x^2 → let $u = x^2$

$du = d(x^2) = 2x\,dx$

$2x\,dx = du$

$\int \cos u\,du = \sin u + C$

$= \sin(x^2) + C$

$(\sin(x^2))' = \cos(x^2)\cdot 2x$ ✓

Undo → u-substitution

SHORT A CONSTANT: FIX IT

Short a constant → fix it

$\int e^{x^2} x\,dx$

Inside: x^2 → $u = x^2$, $du = 2x\,dx$

Top has $x\,dx$, not $2x\,dx$

$x\,dx = \frac{1}{2}du$

$\int e^u \cdot \frac{1}{2}du = \frac{1}{2}e^u + C$

$= \frac{1}{2}e^{x^2} + C$

$\int \cos 3x\,dx$ → 3x inside

$u = 3x$, $du = 3\,dx$ → $dx = \frac{1}{3}du$

$\frac{1}{3}\sin u + C = \frac{1}{3}\sin 3x + C$

TOP = DERIVATIVE OF BOTTOM: LN

Top = derivative of bottom → ln

$\int \frac{2x}{x^2+5}\,dx$

No inside function → try u = bottom

$u = x^2 + 5$ → $du = 2x\,dx$ → the top

$\int \frac{du}{u} = \ln|u| + C$

$= \ln(x^2+5) + C$, since $x^2 + 5 > 0$

Top = derivative of bottom →

$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

AN EXTRA X LEFT OVER

An extra x left over

$\int x\sqrt{x-1}\,dx$

Under the root: $u = x - 1$, $du = dx$

Extra x → $x = u + 1$

$$\int (u+1)u^{\frac{1}{2}}\,du = \int \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)du$$

$$= \frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + C$$

DEFINITE INTEGRAL: CHANGE THE ENDS

Definite → change the ends

$$\int_0^1 2x(x^2+1)^3\,dx$$

$u = x^2+1$, $du = 2x\,dx$

0 and 1 belong to x → change the ends

$x = 0 \to u = 1$, $x = 1 \to u = 2$

$$= \int_1^2 u^3\,du = \left[\frac{u^4}{4}\right]_1^2$$

$$= \frac{16}{4} - \frac{1}{4} = \frac{15}{4}$$

New ends → stay in u

or back to x → old ends. Never both.

PROBLEM 1 OF 3

Change the ends

$\int_1^e \frac{(\ln x)^2}{x}\,dx$ is

(A) $\frac{1}{3}$ (B) $\frac{e^3-1}{3}$ (C) $\frac{e}{3}$ (D) 1

Solution Inside, then new ends

Plan: inside → u → new ends

$\ln x$ inside; $(\ln x)' = \frac{1}{x}$ is there

$u = \ln x$, $du = \frac{1}{x}\,dx$

$x = 1 \to u = 0$, $x = e \to u = 1$

$$\int_0^1 u^2\,du = \left[\frac{u^3}{3}\right]_0^1 = \frac{1}{3}$$

So (A)

Workbook: Problem 4

old ends on u

✗ Common mistake

$$\int_1^e u^2\,du$$

$$= \frac{e^3-1}{3} \to (B)$$

✓ Do this

$$\int_0^1 u^2\,du = \frac{1}{3}$$

u → new ends

PROBLEM 2 OF 3

A basic rule in disguise

Find $\int \frac{e^x}{1+e^{2x}}\,dx$.

Solution The bottom fails, look inside

Plan: u = bottom? If not → look inside

$u = 1+e^{2x} \to du = 2e^{2x}\,dx$ ✗

$e^{2x} = (e^x)^2$

$u = e^x$, $du = e^x\,dx$

$$\int \frac{du}{1+u^2} = \arctan u + C$$

$$= \arctan(e^x) + C$$

Look for $1+(\cdots)^2$

PROBLEM 3 OF 3

No formula, only a table

f is differentiable, and f' is continuous. The table gives values of f.

Find $\int_0^2 f'(3x+1)\,dx$.

x	0	1	2	7
$f(x)$	3	5	6	−1

Solution Substitute, then read the table

Plan: inside $3x+1 \to u \to$ new ends $\to$ table

$u = 3x+1, \quad du = 3\,dx \to dx = \frac{1}{3}du$

$x = 0 \to u = 1, \quad x = 2 \to u = 7$

$\frac{1}{3}\int_1^7 f'(u)\,du = \frac{1}{3}(f(7) - f(1))$

$f(7) = -1,\ f(1) = 5$

$= \frac{1}{3}(-1-5) = \frac{-6}{3} = -2$

$\frac{1}{3}(6-3) = 1$ ✗

Workbook: Problem 6

WRAP-UP

Undo the chain rule?

Yes → inside function → u

Top = bottom′ → ln

Definite → u's ends

Next → 6.10: algebra first

Next: the workbook for this topic (link in the video description). Stuck? Ask in the comments.