Lecture Notes · Unit 6 · Lecture 4 AP CED 6.4

FTC and accumulation functions

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INTRO: HOW FAST DOES THE PILE GROW?

Derivative of an Integral AP Calculus 6.4

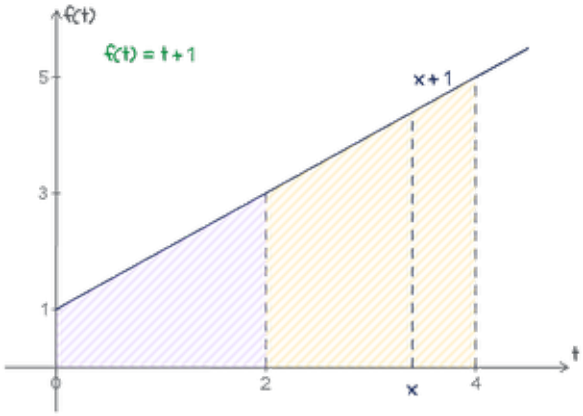
Rate f(t) piled up to time x → F(x)

x moves → F grows. How fast?

$\frac{d}{dx}\int_2^x \sqrt{t^2+5}\,dt \rightarrow ?$

No integrating → one line

THE AREA SO FAR



Rate f(t) = t + 1, start at 0

Stop at 2 → $\frac{1+3}{2}\cdot 2 = 4$

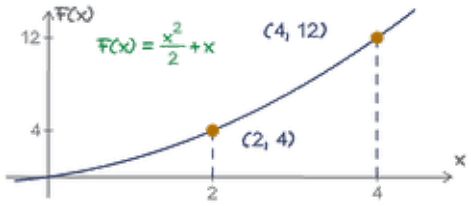
out to 4 → $+\frac{3+5}{2}\cdot 2 = 8 \rightarrow 12$

Stop at x → $\frac{1+(x+1)}{2}\cdot x$

$= \frac{x^2}{2}+x$

area so far, 0 to x → F(x)

$F(x) = \int_0^x f(t)\,dt$

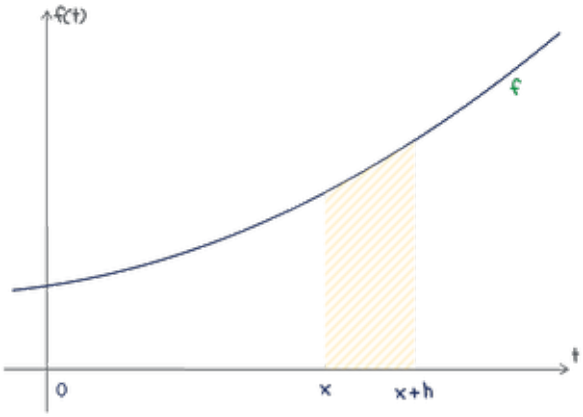


area so far → accumulation function

F'(x) = x + 1 = f(x)

WHY F PRIME EQUALS F

Why F'(x) = f(x)

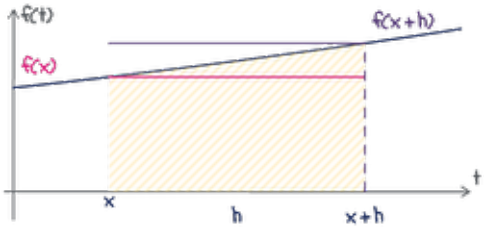


$F(x) = \int_0^x f(t)\,dt$ → purple area

F(x+h) → purple area + 1 new strip

F(x+h) − F(x) → strip only

thin strip → width h



$f(x)\cdot h \le F(x+h)-F(x) \le f(x+h)\cdot h$

$\div h \rightarrow f(x) \le \frac{F(x+h)-F(x)}{h} \le f(x+h)$

middle, $h \to 0 \to F'(x)$

f continuous $\to f(x+h) \to f(x)$

Squeezed $\to F'(x) = f(x)$

$F'(x)$ = edge height

ANY STARTING POINT A

Starting at a

0 to $x \to F(x)$

0 to $a \to F(a)$, a fixed number

$$\int_a^x f(t)\,dt = F(x) - F(a)$$

t? $x \to$ moving end

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) - 0$$

If f continuous $\to \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

any start a $\to$ Fundamental Theorem of Calculus

No formula $\to \frac{d}{dx}\int_0^x e^{-t^2}\,dt = e^{-x^2}$

WHEN THE TOP END IS X SQUARED

When the top end moves

$$\frac{d}{dx}\int_0^{x^2} (t+1)\,dt \to ?$$

Names first $\to f(t) = t+1$, top $= x^2$

$$\int_0^{x^2} (t+1)\,dt = F(x^2) - F(0)$$

$\frac{d}{dx}$ both sides $\to F'(x^2) \cdot 2x - 0$

$$= f(x^2) \cdot 2x = (x^2+1) \cdot 2x$$

Check $\to F(x^2) = \frac{x^4}{2} + x^2 \to 2x^3 + 2x$ ✓

$f(\text{top}) \times$ top derivative

$$\int_{v(x)}^{u(x)} f(t)\,dt = F(u) - F(v)$$

$$\frac{d}{dx} \to f(u)\,u' - f(v)\,v'$$

PROBLEM 1 OF 2

An accumulation function: values, tangent line

Let $F(x) = \int_2^x \sqrt{t^2+5}\,dt$. Find $F(2)$, $F'(2)$, a tangent line, and $F''(2)$.

(a) Find $F(2)$ and $F'(2)$. Write an equation for the tangent line at $x = 2$.

(b) Find $F''(2)$.

Solution (a) Value, slope, line

(a)

Plan $\to$ ends, $F' \to$ point + slope

$$F(2) = \int_2^2 \sqrt{t^2+5}\,dt = 0$$

$\sqrt{t^2+5}$ continuous $\to$ FTC

$$F'(x) = \sqrt{x^2+5}$$

$$F'(2) = \sqrt{4+5} = \sqrt{9} = 3$$

$(2, 0)$, $m = 3$

$$y = 0 + 3(x-2)$$

Solution (b) The second derivative

(b)

$$F'' = (F')' \to F'(x) = (x^2+5)^{\frac{1}{2}}$$

$$F''(x) = \frac{1}{2}(x^2+5)^{-\frac{1}{2}} \cdot 2x$$

$$= \frac{x}{\sqrt{x^2+5}}$$

$$F''(2) = \frac{2}{\sqrt{9}} = \frac{2}{3}$$

Workbook $\to$ Problems 2, 4

PROBLEM 2 OF 2

A moving end

Let $g(x)=\int_1^{x^2}\frac{1}{1+t}\,dt$. What is $g'(1)$?

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) $\ln 2$

Solution Name f, then both sides

Plan: $f \to F(x^2) - F(1) \to \frac{d}{dx}$

$f(t)=\frac{1}{1+t}$, $u=x^2$

$F(u)=\int_0^u f(t)\,dt$, $g(x)=F(x^2)-F(1)$

$g'(x)=f(x^2)\cdot 2x - 0$

$=\frac{1}{1+x^2}\cdot 2x$

$g'(1)=\frac{1}{2}\cdot 2 = 1$

Solution Name f, then both sides

So (B)

(A) $\frac{1}{2}$ → forgot $\cdot 2x$

workbook $\sin x$, $3x$, x^3 → 1, 3, 5, 6

Trap: the leftover t

X Common mistake

$F(x)=\int_2^x \sqrt{t^2+5}\,dt$

$F'(x)=\sqrt{t^2+5}$

✓ Do this

$t \to x$

$F'(x)=\sqrt{x^2+5}$

Trap: integrating first

X Common mistake

$\frac{d}{dx}\int_2^x \sqrt{t^2+5}\,dt$

integrate $\sqrt{t^2+5}$ first?

✓ Do this

f continuous → FTC

$F'(x)=\sqrt{x^2+5}$

Trap: the missing 2x

X Common mistake

$\frac{d}{dx}\int_0^{x^2}(t+1)\,dt$

$=x^2+1$

✓ Do this

top $=x^2$ → times $(x^2)'=2x$

$=(x^2+1)\cdot 2x$

WRAP-UP

How fast does F grow?

Rate of F = height of f at the edge

If f continuous → $F'(x)=f(x)$, any start a

Top moves → x the top's derivative

Next → 6.5 → what f says about F

Next: the workbook for this topic (link in the video description). Stuck? Ask in the comments.