6.4 / The Fundamental Theorem of Calculus and Accumulation Functions

Watch the lesson ➤

F(x) = the area so far, F' = f, and a moving upper end with the chain rule

Learning goal

Use $F(x) = \int_a^x f(t)\,dt$ as the amount so far, write $F'(x) = f(x)$ (the Fundamental Theorem), and differentiate with a moving upper end.

Key ideas

01 The amount so far: $F(x) = \int_0^x f(t)\,dt$. With $f(t) = t + 1$: $F(2) = 4$, $F(4) = 12$, and $F(x) = \frac{x^2}{2} + x$. x moves, t fills in: an **accumulation function**.

02 One thin strip more: $F(x + h) - F(x)$ is the strip of width h. f rising: $f(x) \cdot h \le F(x + h) - F(x) \le f(x + h) \cdot h$. Divide by h; $h \to 0$ and f continuous: $F'(x) = f(x)$.

03 f continuous: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$, the **Fundamental Theorem** of Calculus. Write $F'(x) = f(x)$ first.

04 Area 0 to a is $F(a)$, so the area a to x is $\int_a^x f(t)\,dt = F(x) - F(a)$ (same number with t or x inside: x is the end, so t). $\frac{d}{dx}$: $F'(x) - 0$.

05 A moving end: $\int_0^{x^2} (t + 1)\,dt = F(x^2) - F(0)$, so $\frac{d}{dx} = f(x^2) \cdot 2x = (x^2 + 1) \cdot 2x$. In general $\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = f(u(x))\,u'(x)$.

Remember

accumulation function
$F(x) = \int_a^x f(t)\,dt$: the signed area under f from a to x.

Fundamental Theorem of Calculus
For f continuous: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

Class notes

6.4 / Write it clearly

What earns the point on the AP Exam, and the mistakes that lose it.

Common mistakes

The mistake code is used in the answers and in My Mistakes

01 t left in, or the long way

✗ $F'(x) = \sqrt{t^2 + 5}$; or work out the integral first.

✓ Put x where t was: $F'(x) = \sqrt{x^2 + 5}$. No integrating: one line.

6.4 / Guided examples

Watch, pause, and write the reasoning. Full solutions are in the video.

Problem types

Read the problem, name the type, then use its method

Type	Method	Problems
g' = f	Write g'(x) = f(x) first; values of g are areas (same ends give 0).	E01 · Practice 2
A moving upper end	f at the upper end, times the derivative of the upper end.	E02 · Practice 1, 3, 4, 5, 6

Example 01

Video · E01

Let $F(x) = \int_2^x \sqrt{t^2 + 5}\, dt$. Find $F(2)$, $F'(2)$, a tangent line, and $F''(2)$.
(a) Find $F(2)$ and $F'(2)$. Write the tangent line at $x = 2$.
(b) Find $F''(2)$.

> Same ends give 0. Write $g'(x) = f(x)$ first.

Example 02

Video · E02

$g(x) = \int_1^{x^2} \frac{1}{1+t}\, dt$. $g'(1)$ is

A. $\frac{1}{2}$
B. 1
C. 2
D. $\ln 2$

> Name $f(t) = \frac{1}{1+t}$; then $h(x) = F(x^2) - F(1)$.

VIDEO COMPLETE Check your example answers at the end of this booklet.

6.4 / Example answers

The Fundamental Theorem of Calculus and Accumulation Functions

E01 $g(2) = \int_2^2 \sqrt{t^2+5}\,dt = 0$

$\sqrt{t^2+5}$ is continuous, so $g'(x) = \sqrt{x^2+5}$ and $g'(2) = 3$.

(a) Tangent line: $y = 0 + 3(x-2)$, so $y = 3(x-2)$.

$g''(x) = \dfrac{x}{\sqrt{x^2+5}} \;\rightarrow\; g''(2) = \dfrac{2}{3}$

E02 (B) $h(x) = F(x^2) - F(1), \quad F' = f$

$h'(x) = f(x^2)\cdot 2x = \dfrac{1}{1+x^2}\cdot 2x$

$h'(1) = \dfrac{1}{2}\cdot 2 = 1$

Answer B.