Lecture Notes · Unit 6 · Lecture 3 AP CED 6.3

Riemann sums, sigma, definite integral

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INTRO: WHERE DOES THE SUM GO?

Riemann Sums → Definite Integral AP Calculus 6.3

Slice thinner → where does the sum go?

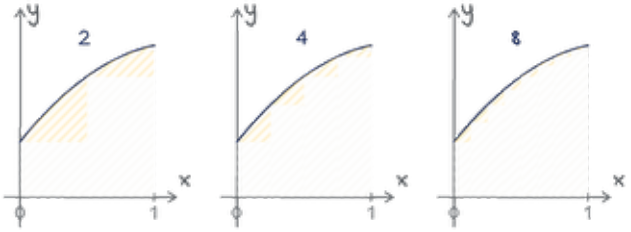
Today → limit of a sum ⇔ definite integral

Why it's exact → a picture

MORE PIECES, CLOSER TO THE AREA

More pieces

[0, 1] → 2 right ends



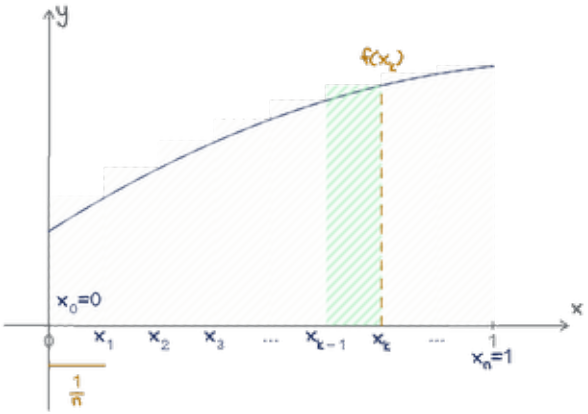
4, 8 pieces → each pink bit smaller

Left ends → yellow gaps shrink too

More pieces → closer to the area

N PIECES, AND SIGMA

n pieces on [0, 1]



First → n pieces, then n → ∞

width = $\frac{1}{n} = \Delta x$

k steps of $\frac{1}{n}$ → $x_k = \frac{k}{n}$

$f(x_k) = f\left(\frac{k}{n}\right)$

$S = \frac{1}{n}f(x_1) + \frac{1}{n}f(x_2) + \cdots + \frac{1}{n}f(x_n)$

add up, k = 1 to n → $\sum_{k=1}^{n}$

$S = \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n}$

n pieces → S ≈ area

N TO INFINITY: THE DEFINITE INTEGRAL

Let n go to infinity

$\lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n} = \int_0^1 f(x)\,dx$

Why 0 and 1?

$x_k = \frac{k}{n}$, $k = 1 \rightarrow x_1 = \frac{1}{n}$ → 0 start

$k = 2 \rightarrow x_2 = \frac{2}{n}$

$k = n \rightarrow x_n = \frac{n}{n} = 1$ end

$x_2 - x_1 = \frac{2}{n} - \frac{1}{n} = \frac{1}{n} = \Delta x$

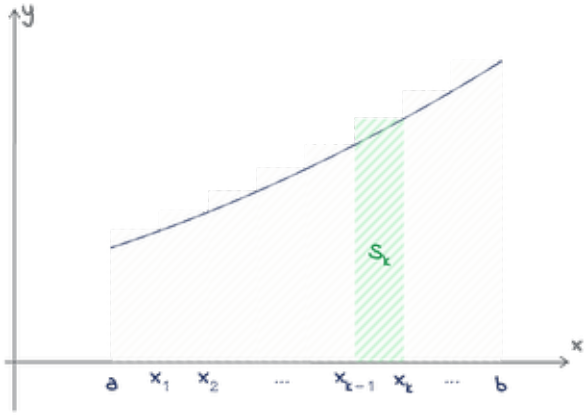
$n \to \infty \to \Delta x \to 0 \to$ write dx for Δx

Match $\to \frac{k}{n} \to x$, $\frac{1}{n} \to dx$, $k=1 \to 0$, $k=n \to 1$

definite integral = exact area

$f(x)\,dx \to$ area of a very thin strip

ANY INTERVAL [A, B]



$[0, 1] \to [a, b]$, n pieces

$\Delta x = x_k - x_{k-1} = \frac{b-a}{n}$

$a \to k$ steps of $\frac{b-a}{n}$

$x_k = a + \frac{b-a}{n}k$

$S_k = f\left(a + \frac{b-a}{n}k\right)\frac{b-a}{n}$

$S_n = \sum_{k=1}^{n} S_k$

f continuous: $[a, b]$, $n \to \infty$

$\lim_{n \to \infty} S_n = \int_a^b f(x)\,dx$

signed $\to$ below the axis counts negative

ONE LIMIT, THREE INTEGRALS

Three ways $\to$

$$\lim_{n \to \infty} \sum_{k=1}^{n} f\left(a + \frac{b-a}{n}k\right)\frac{b-a}{n}$$

way 1 $\to x_k = a + \frac{b-a}{n}k$

$k = 1 \to a + \frac{b-a}{n}$, $n \to \infty \to a + 0 = a$

$k = 2 \to a + \frac{b-a}{n} \cdot 2$

$k = n \to a + \frac{b-a}{n} \cdot n = b$

$x_2 - x_1 = \frac{b-a}{n} = \Delta x \to dx$

$\to \int_a^b f(x)\,dx$

Rule $\to$ the part with k is x

$k \to 1$ start, n end, $x_2 - x_1 \to dx$

way 2 $\to x_k = \frac{b-a}{n}k$

$x_1 = \frac{b-a}{n} \to 0$, $x_n = b - a$

$\to \int_0^{b-a} f(a + x)\,dx$

way 3 $\to$ only $\frac{k}{n}$ is $x \to 0$ to 1, $dx = \frac{1}{n}$

$\to (b-a)\int_0^1 f(a + (b-a)x)\,dx$

1 limit $\to$ 3 integrals, same area

READING A SUM BACK: UP TO N - 1

Read it back

$$\lim_{n \to \infty} \sum_{k=1}^{n-1} \left(1 + \frac{2k}{n}\right)^2 \frac{2}{n}$$

$x_k = 1 + \frac{2k}{n}$

$k = 1 \to 1 + \frac{2}{n} \to 1$

$k = n - 1 \to 1 + \frac{2(n-1)}{n} = 3 - \frac{2}{n} \to 3$

$x_2 - x_1 = \frac{2}{n} \to dx$

$f(x) = x^2 \to \int_1^3 x^2\,dx$

$\frac{2}{n} \cdot 3^2 = \frac{18}{n} \to 0$

way 2 $\to \int_0^2 (1 + x)^2\,dx$, same area

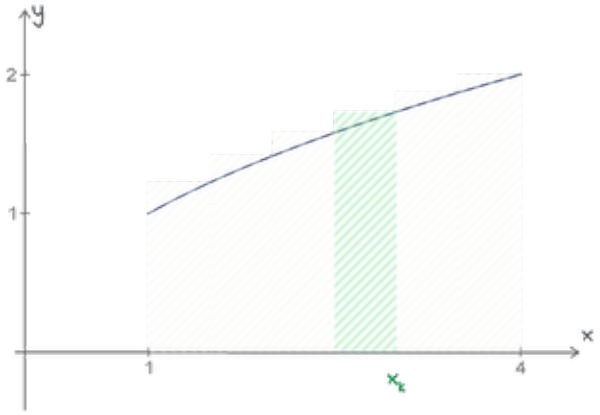
PROBLEM 1 OF 2

An integral as a sum, and as a limit

Consider $\int_1^4 \sqrt{x}\,dx$, first as a sum with Σ, then as a limit.

(a) Write a right sum with n equal pieces, using Σ.

(b) Write the integral as the limit of that sum.



Solution (a) The right sum with sigma

(a)

Plan → Δx → x_k → $f(x_k)$

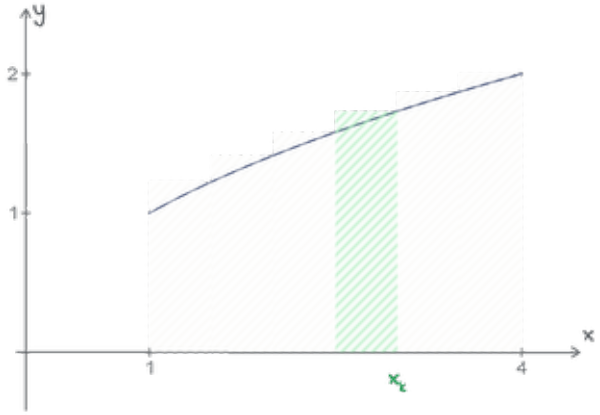
$$\Delta x = \frac{4-1}{n} = \frac{3}{n}$$

$$x_k = a + k\Delta x = 1 + \frac{3k}{n}$$

$$f(x_k) = \sqrt{1 + \frac{3k}{n}}$$

$$\sum_{k=1}^{n} \sqrt{1 + \frac{3k}{n}} \cdot \frac{3}{n} \approx \int_1^4 \sqrt{x}\,dx$$

Solution (b) The integral as a limit



(b)

$$\int_1^4 \sqrt{x}\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1 + \frac{3k}{n}} \cdot \frac{3}{n}$$

$$x_1 = 1 + \frac{3}{n} \to 1,\ x_n = 1 + 3 = 4$$

Workbook → Problems 1, 4(b), 6 → Δx, x_k, f

PROBLEM 2 OF 2

Which integral is it?

$\lim_{n\to\infty} \sum_{k=1}^{n} (2 + \frac{5k}{n})^3 \frac{5}{n}$ is equal to

(A) $\int_0^5 x^3\,dx$

(B) $\int_2^7 x^3\,dx$

(C) $\int_2^7 (2+x)^3\,dx$

(D) $5\int_2^7 x^3\,dx$

Solution Read the limit as an integral

Plan → x_1 → a, x_n → b, $x_2 - x_1$ → dx, then f

$$x_k = 2 + \frac{5k}{n}$$

$$k = 1 \to 2 + \frac{5}{n} \to 2 = a$$

$$k = n \to 2 + 5 = 7 = b$$

$$x_2 - x_1 = \frac{5}{n} \to dx$$

$f(x) = x^3$

$\int_2^7 x^3\,dx \rightarrow$ (B)

Solution Read the limit as an integral

way 2 → $\int_0^5 (2+x)^3\,dx$ also true, not listed

Workbook → Problems 2, 3, 5

Careful

X Common mistake

(A) start 2 dropped

(C) start 2 used twice

(D) width 5 used twice

✓ Do this

Same x → its ends and dx

$x_k = 2 + \frac{5k}{n} \rightarrow$ (B)

WRAP-UP

Where does the sum go?

$\lim_{n\to\infty}$ of sum → exact signed area

Read → x, dx, $x_1 \rightarrow a$, $x_n = b$

Next → 6.4: let the end move

Next: the workbook for this topic (link in the video description). Stuck? Ask in the comments.