6.3 / Riemann Sums, Summation Notation, and Definite Integral Notation

Watch the lesson ➤

The right sum with sigma, its limit as the definite integral, and reading a limit back

Learning goal

Write a definite integral as the limit of a Riemann sum with Σ, and read a limit of a sum back as an integral (Δx, a, b, f).

Key ideas

01 More pieces: the over-rectangles and the under-rectangles both close in on the curve. The sum goes to the **area**.

02 n equal pieces of $[a, b]$: $\Delta x = \frac{b-a}{n}$, and $x_k = a + \frac{b-a}{n}k$ (a plus Δx, k times). $k = 1$: $x_1 = a + \frac{b-a}{n} \to a$; $k = n$: $x_n = b$; $x_2 - x_1 = \Delta x$.

03 Right sum: $\displaystyle\sum_{k=1}^{n} f\left(a + \frac{b-a}{n}k\right)\frac{b-a}{n}$. f continuous on $[a, b]$: $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\,\Delta x = \int_a^b f(x)\,dx$, the **definite integral**.

04 Read it back: $\lim \sum \left(1 + \frac{2k}{n}\right)^2 \frac{2}{n}$. $\Delta x = \frac{2}{n}$; $x_k = 1 + \frac{2k}{n}$ starts at 1 and ends at 3; $x_k \to x$: $\int_1^3 x^2\,dx$. Or $x_k = \frac{2k}{n}$ from 0 to 2: $\int_0^2 (1+x)^2\,dx$.

Remember

summation notation

$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$.

definite integral

$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\,\Delta x$, for f continuous on $[a, b]$.

Class notes

6.3 / Write it clearly

What earns the point on the AP Exam, and the mistakes that lose it.

Common mistakes

The mistake code is used in the answers and in My Mistakes

01 The start left out, or added twice

✗ $\lim \sum \left(2 + \frac{5k}{n}\right)^3 \frac{5}{n} = \int_0^5 x^3\,dx$, or $\int_2^7 (2+x)^3\,dx$.

✓ $x_k = a + k\,\Delta x$ gives $a = 2$, $b = 7$; f is what is done to x_k: x^3. So $\int_2^7 x^3\,dx$.

6.3 / Guided examples

Watch, pause, and write the reasoning. Full solutions are in the video.

Problem types

Read the problem, name the type, then use its method

Type	Method	Problems
An integral as a limit of a right sum	$\Delta x = \frac{b-a}{n}$, $x_k = a + k\,\Delta x$, then the sum of $f(x_k)\,\Delta x$ and its limit.	E01 · Practice 1, 4, 6
Read a limit back	$\Delta x \to dx$ and $b - a$; $x_k \to x$, its start a ($k = 1$, $n \to \infty$) and its end b ($k = n$); then f.	E02 · Practice 2, 3, 5

Example 01

Video · E01

Consider $\int_1^4 \sqrt{x}\,dx$: as a sum with Σ, then as a limit.
(a) Right sum, n equal pieces, with Σ
(b) The integral as a limit of that sum

> Δx first, then $x_k = a + k\,\Delta x$, then $f(x_k)$.

Example 02

Video · E02

$\lim_{n\to\infty} \sum_{k=1}^{n} \left(2 + \frac{5k}{n}\right)^3 \frac{5}{n}$ is equal to

A. $\int_0^5 x^3\,dx$
B. $\int_2^7 x^3\,dx$
C. $\int_2^7 (2+x)^3\,dx$
D. $5\int_2^7 x^3\,dx$

> Read Δx, then a from x_k, then $b = a + (b - a)$, then f.

VIDEO COMPLETE Check your example answers at the end of this booklet.

6.3 / Example answers

Riemann Sums, Summation Notation, and Definite Integral Notation

E01 $\Delta x = \dfrac{4-1}{n} = \dfrac{3}{n}, \quad x_k = 1 + \dfrac{3k}{n}$

(a) $\displaystyle\sum_{k=1}^{n} \sqrt{1 + \frac{3k}{n}} \cdot \frac{3}{n}$

(b) $\displaystyle\int_1^4 \sqrt{x}\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1 + \frac{3k}{n}} \cdot \frac{3}{n}$

Check: $a = 1, b = 1 + 3 = 4$.

E02 (B) $x_1 = 2 + \dfrac{5}{n} \;\to\; 2 = a, \quad x_n = 2 + 5 = 7 = b, \quad \Delta x = \dfrac{5}{n} \;\to\; dx$

x_k is cubed, so $f(x) = x^3$: $\int_2^7 x^3\,dx$.

Answer B.