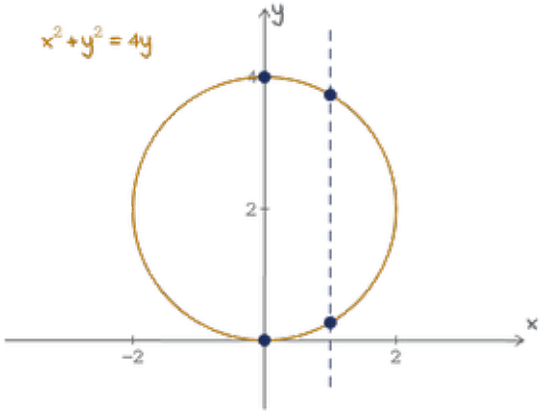
Lecture Notes · Unit 5 · Lecture 12 AP CED 5.12

Implicit relations

Tap a blue play button next to a heading to open that part of the video. ▶ Watch the video

INTRO: A CIRCLE YOU CAN'T WRITE AS Y = F(X)

Implicit Relations AP Calculus 5.12



$x^2+y^2=4y$ → center $(0,2)$, radius 2

See it → top $(0,4)$; bottom $(0,0)$

Equation only? one x → two y's

Today → flat? vertical? top or bottom?

FLAT: HORIZONTAL TANGENTS

Where is it flat?

d each variable → $2x\,dx+2y\,dy=4\,dy$

$\div dx \rightarrow y'=\frac{x}{2-y}$

flat → slope 0 → top = 0, bottom ≠ 0

top x = 0 → a line, not a point

back into curve → $y^2=4y \rightarrow y(y-4)=0$

$y=0,\ 4$; bottom $2-y \rightarrow 2,\ -2 \neq 0$

Flat at $(0,0)$, $(0,4)$

VERTICAL TANGENTS

Where is the tangent vertical?

vertical → bottom $2-y=0$, top $x\neq 0$

$y=2$

$x^2+4=8 \rightarrow x=\pm 2$

$(\pm 2, 2)$: $x\neq 0$ → vertical

back in → $x^2<0$? → no such point

flat, vertical → critical points

TOP OR BOTTOM: THE SECOND DERIVATIVE

Top or bottom?

x = 0: both flat. Top or bottom?

y′ has a y → x alone can't tell

$u=x,\ u'=1,\quad v=2-y \rightarrow v'=-y'$

$$y''=\frac{1\cdot(2-y)-x(-y')}{(2-y)^2}$$

flat point → $x=0,\ y'=0 \rightarrow x\cdot(-y')=0$

$$y''=\frac{2-y}{(2-y)^2}=\frac{1}{2-y}$$

THE SECOND DERIVATIVE TEST ON A CURVE

The sign decides

$y'=0 \rightarrow y''=\frac{1}{2-y}$

$(0,4) \to \frac{1}{2-4} = -\frac{1}{2} < 0$

→ concave down → relative max

$(0,0) \to \frac{1}{2} > 0$ → concave up → relative min

→ Second Derivative Test

$y'' = 0$ → can't tell → sign of y'

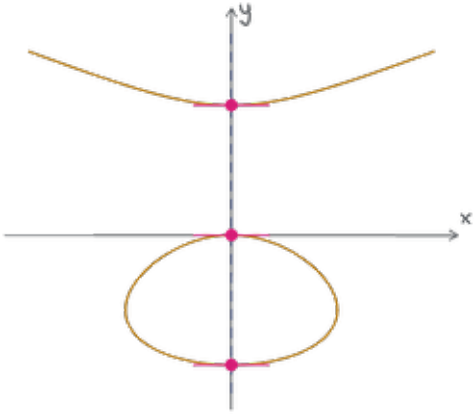
PROBLEM 1 OF 2

Flat points, max or min

The curve $y^3 - 3y = x^2$ has $\frac{dy}{dx} = \frac{2x}{3y^2 - 3}$.

(a) Find every point where the tangent line is horizontal.

(b) At each point, is there a relative max or a relative min? Use the second derivative.



Solution (a) Horizontal tangents

(a)

Plan: top = 0 → back into curve → check bottom

$2x = 0 \to x = 0$

$x = 0$ → a line → $y^3 - 3y = 0$

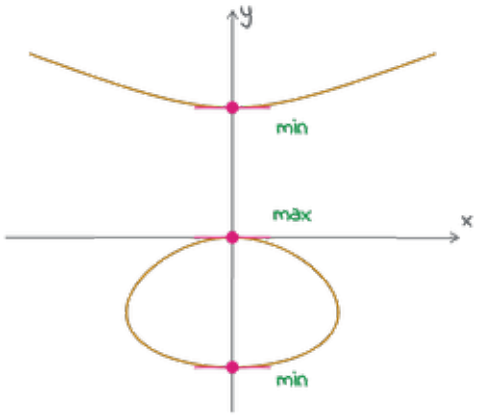
$y(y^2 - 3) = 0 \to y = 0, \pm\sqrt{3}$

bottom $3y^2 - 3 \to -3, 6, 6 \neq 0$

Flat at $(0,0), (0,\sqrt{3}), (0,-\sqrt{3})$ 3 points, not 1

Your turn → (b): $\frac{d^2y}{dx^2}$ at $x = 0$, $y' = 0$

Solution (b) Max or min



(b)

$u = 2x$, $v = 3y^2 - 3$ → $u' = 2$, $v' = 6yy'$

$$\frac{d^2y}{dx^2} = \frac{2(3y^2 - 3) - 2x \cdot 6yy'}{(3y^2 - 3)^2}$$

$x = 0$, $y' = 0$ → $\frac{d^2y}{dx^2} = \frac{2}{3y^2 - 3}$

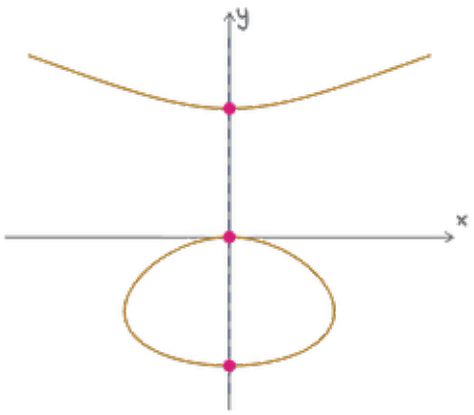
$(0,0) \to \frac{2}{-3} < 0$ → rel. max

$(0, \pm\sqrt{3}) \to \frac{2}{6} = \frac{1}{3} > 0$ → rel. min

$\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} < 0$ at $(0,0)$ → rel. max

Workbook → back into the curve first

$x = 0$ is a line, not a point



✗ Common mistake

$2x = 0 \to x = 0 \to (0, 0)$

✓ Do this

$x = 0 \to$ a line

$y^3 - 3y = 0 \to y = 0, \pm\sqrt{3} \to$ 3 points

PROBLEM 2 OF 2

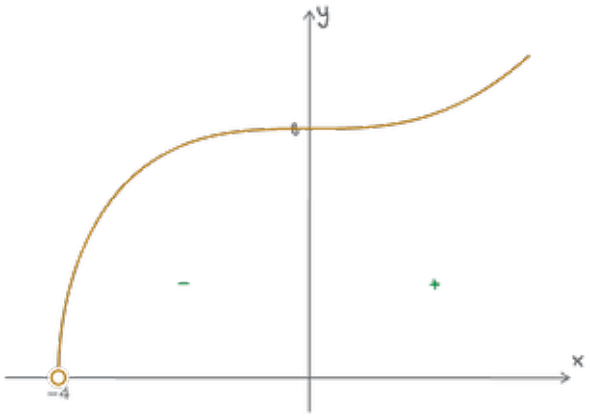
Concavity on a curve

The curve $y^2 = x^3 + 64$, $y > 0$, has $\dfrac{dy}{dx} = \dfrac{3x^2}{2y}$,

$\dfrac{d^2y}{dx^2} = \dfrac{3x(x^3 + 256)}{4y^3}$

(a) Where is the curve concave up, and where is it concave down?

(b) At the horizontal tangent, is there a rel. max, a rel. min, or neither? Is it an inflection point?



Solution (a) Concavity

(a)

Plan: which x? → sign of each piece

$y > 0 \to x^3 + 64 > 0 \to x > -4$

$x^3 > -64 \to x^3 + 256 > 0$

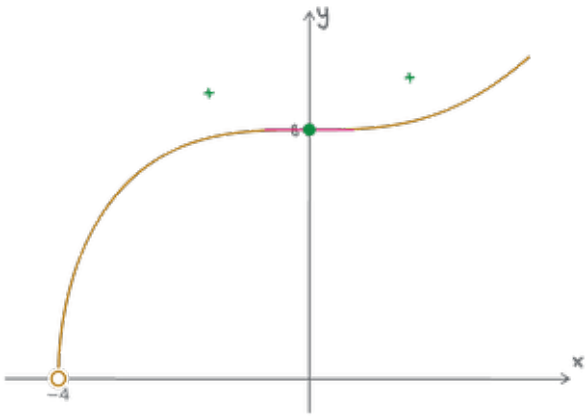
$y > 0 \to 4y^3 > 0$

sign of y'' = sign of x

$(-4, 0)$: down $(0, \infty)$: up

Your turn → the flat point first

Solution (b) The test can't tell



(b)

top $3x^2 = 0 \to x = 0 \to y^2 = 64$

$y > 0 \to y = 8$; bottom $16 \neq 0 \to (0, 8)$

$\dfrac{d^2y}{dx^2}$ at $(0, 8) \to 0 \to$ the test can't tell

$3x^2 \geq 0$, $2y > 0$

+ before 0, + after → neither

Down → up → $(0, 8)$ inflection

y': relative min at $x = 0$

Workbook → concavity on a circle

WRAP-UP

Implicit curves: the plan

Flat → top = 0, bottom ≠ 0

1 input x → 2 or 3 points

vertical → bottom = 0, top ≠ 0

Flat point → $y' = 0$ (here $x = 0$) → sign of y'' decides

$y'' = 0$ → back to sign of y'

Next → Unit 6 · adding up change

Next: the workbook for this topic (link in the video description). Stuck? Ask in the comments.