5.12 / Exploring Behaviors of Implicit Relations

Watch the lesson ➤

Tops, bottoms, and the sign of the second derivative

Learning goal

Find horizontal and vertical tangents of an implicit curve, put each point back into the curve, and use the sign of $\frac{d^2y}{dx^2}$ to tell a high point from a low point.

Key ideas

01 Flat: $\frac{dy}{dx} = 0$. For $x^2 + y^2 = 4y$: $y' = \frac{x}{2-y}$, the top $= 0$ gives $x = 0$; back into the curve, $y^2 = 4y$, $y = 0,\ 4$. The bottom $2 - y \neq 0$: $(0, 0)$ and $(0, 4)$.

02 Upright: the bottom $= 0$ and the top $\neq 0$. $2 - y = 0$ gives $y = 2$; into the curve, $x^2 + 4 = 8$, $x = \pm 2$: vertical tangents at $(\pm 2,\ 2)$.

03 Always put the point back into the **curve**.

04 High or low: with the quotient rule, $y'' = \frac{(2-y)\cdot 1 - x\cdot(-y')}{(2-y)^2}$; at $x = 0$, $y' = 0$, $y'' = \frac{1}{2-y}$. $(0, 4)$: $y'' < 0$, a relative max. $(0, 0)$: $y'' > 0$, a relative min.

Remember

horizontal tangent

$\frac{dy}{dx} = 0$: top $= 0$, bottom $\neq 0$.

vertical tangent

The bottom of $\frac{dy}{dx}$ is 0 and the top is not.

Class notes

5.12 / Write it clearly

What earns the point on the AP Exam, and the mistakes that lose it.

Common mistakes

The mistake code is used in the answers and in My Mistakes

01 Never back into the curve

✗ $x = 0$, so the point is $(0, 0)$. Done.

✓ $x = 0$ into the curve gives $y = 0, \pm\sqrt{3}$; with the bottom $\neq 0$, there are three points.

5.12 / Guided examples

Watch, pause, and write the reasoning. Full solutions are in the video.

Problem types

Read the problem, name the type, then use its method

Type	Method	Problems
Flat and upright points	Flat: top $= 0$, bottom $\neq 0$. Upright: bottom $= 0$, top $\neq 0$. Then back into the curve.	Practice 1, 2, 3, 6, 7
The sign of the second derivative	Quotient rule; at a flat point only u'/v is left; its sign $\rightarrow$ max or min.	E01, E02 · Practice 4, 5

Example 01

Video · E01

The curve $y^3 - 3y = x^2$ has $\dfrac{dy}{dx} = \dfrac{2x}{3y^2 - 3}$

(a) Find every point where the tangent line is horizontal.

(b) At each, relative max or relative min? Use the second derivative.

> Top $= 0$, then back into the curve. Then the sign of $\frac{d^2y}{dx^2}$ there.

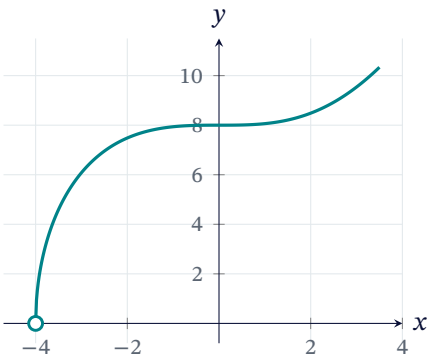
Example 02

Video · E02

The curve $y^2 = x^3 + 64$, $y > 0$, has $\frac{dy}{dx} = \frac{3x^2}{2y}$, $\frac{d^2y}{dx^2} = \frac{3x(x^3+256)}{4y^3}$

(a) Where is the curve concave up? Concave down?

(b) At the horizontal tangent: rel. max, rel. min, or neither? Inflection?



Find which factor decides the sign of $\frac{d^2y}{dx^2}$.

VIDEO COMPLETE Check your example answers at the end of this booklet.

5.12 / Example answers

Exploring Behaviors of Implicit Relations

E01 (a) $(0, 0)$, $(0, \sqrt{3})$, $(0, -\sqrt{3})$; (b) $(0, 0)$ relative max; $(0, \pm\sqrt{3})$ relative min

E02 (a) concave down on $-4 < x < 0$, up on $x > 0$; (b) $(0, 8)$: neither; a point of inflection