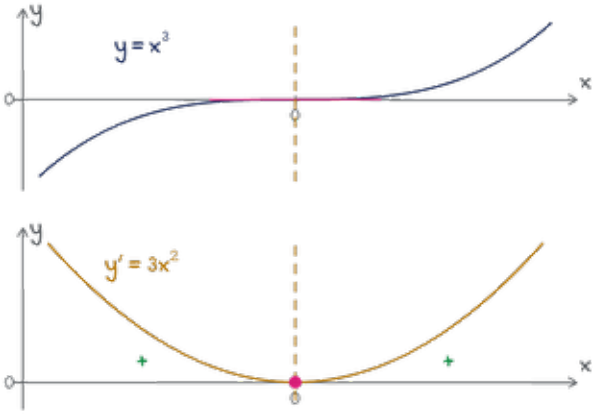
Lecture Notes · Unit 5 · Lecture 7 AP CED 5.7

Second Derivative Test

Tap a blue play button next to a heading to open that part of the video. ▶ Watch the video

INTRO: A FLAT TANGENT

Flat tangent: peak or valley? AP Calculus 5.7



$f'(c) = 0$ → flat tangent at c

Peak? valley? neither?

$f(x) = x^3$ → flat at 0, keeps climbing

$f' = 3x^2$ → + on both sides

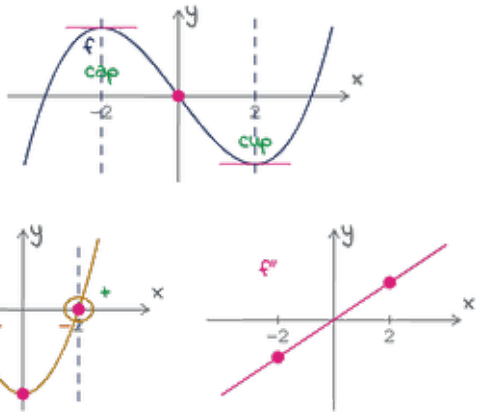
No sign change → neither

one more derivative → does the sign of $f''(c)$ decide?

Today → the bend at a flat spot

IDEA 1: A FLAT SPOT AND THE BEND

A flat spot, and the bend



Idea 1 → flat spot + bend

$f(x) = x^3 - 12x$

$f' = 3x^2 - 12 = 3(x-2)(x+2)$

$f' = 0$ at $x = -2, 2$

f'' = slope of $f' = 6x$

$f''(-2) = -12 < 0$

→ f' falls through 0, + to −

→ rel. max at $x = -2$

$f''(2) = 12 > 0$ → f' rises through 0, − to +

→ rel. min at $x = 2$

1st test → check f' left and right at c

Sign of $f''(c)$ → f' up or down through 0

If $f'(c) = 0$

$f''(c) > 0$ → f' rising through 0 → rel. min

$f''(c) < 0$ → f' falling through 0 → rel. max

Cup → valley · cap → peak

2nd Derivative Test

$f'(c) = 0$ required · no corner

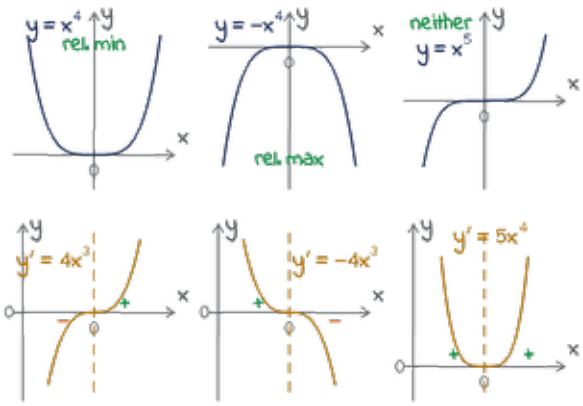
Falling fastest → smallest f' → judge f'

$f''(x) = 6x = 0$ at $x = 0$ · $f'''(x) = 6 > 0$

→ f' lowest at $x = 0$ → f falls fastest

IDEA 2: WHEN F''(C) = 0

When $f''(c) = 0$



Three curves

$y = x^4$ → valley at $x = 0$

$y'' = 12x^2$ → $y''(0) = 0$

$y = -x^4$ → peak

$y'' = -12x^2$ → $y''(0) = 0$

$y = x^5$ → neither

$y'' = 20x^3$ → $y''(0) = 0$

All three → $y'(0) = y''(0) = 0$

$f''(c) = 0$ → no conclusion

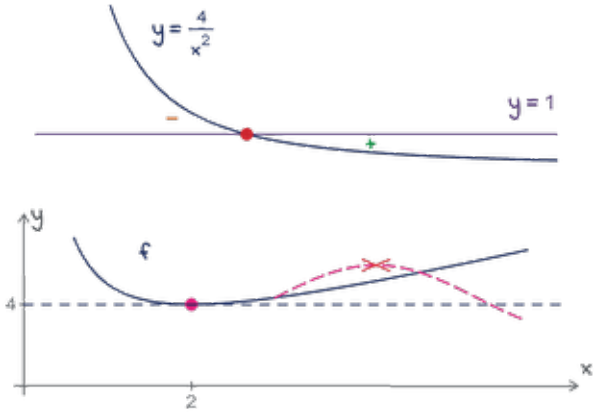
→ check signs of f' (1st test)

$f'' = 0$ → not always inflection (x^4 → cup both sides)

Usually one number → fast · else → 1st test

IDEA 3: ONLY ONE CRITICAL POINT

Only one critical point



Only one critical point?

$f(x) = x + \frac{4}{x}, x > 0$

$f' = 1 - \frac{4}{x^2}$

$y = 1, y = \frac{4}{x^2} \rightarrow x = 2$

left → $f' < 0$ · right → $f' > 0$

$x^2 = 4, x > 0 \rightarrow x = 2$ only

$f'' = \frac{8}{x^3} \rightarrow f''(2) = 1 > 0$

→ rel. min at $x = 2$, $f(2) = 4$

Below 4 elsewhere? → another turn

2nd turn → 2nd critical point → none

→ absolute min = 4

1. f continuous on interval
2. only one critical point

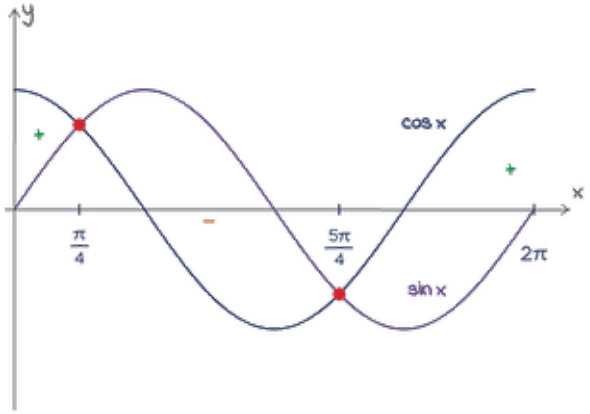
Relative here → absolute

[a, b] compare · no ends: use rule

PROBLEM 1 OF 3

Max and min on $(0, 2\pi)$

Let $f(x) = \sin x + \cos x$ on $(0, 2\pi)$.
Use the Second Derivative Test to find where f has a relative maximum and a relative minimum. Justify your answer.



Solution Where f' = 0

Plan → $f' = 0$, then sign of f'' there

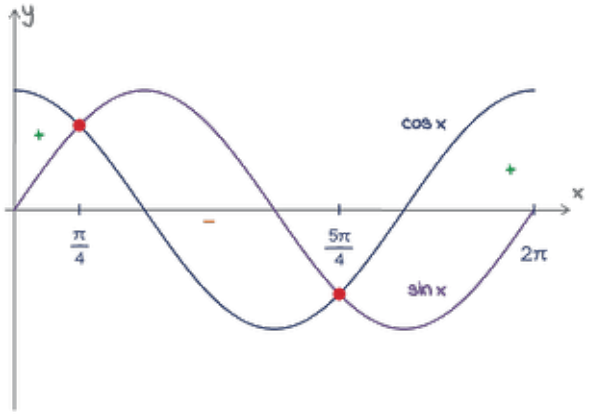
$f' = \cos x - \sin x$ = white − purple

$f' = 0 \rightarrow \cos x = \sin x \rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$

$+ \; - \; + \rightarrow$ max at $\frac{\pi}{4}$, min at $\frac{5\pi}{4}$ (expected)

Asked: test $f'' \rightarrow$ continue

Solution Both facts



$f'' = -\sin x - \cos x$

$f''\left(\frac{\pi}{4}\right) = -\sqrt{2} < 0$

$f'\left(\frac{\pi}{4}\right) = 0$ and $f''\left(\frac{\pi}{4}\right) < 0$

→ rel. max at $x = \frac{\pi}{4}$

$f''\left(\frac{5\pi}{4}\right) = \sqrt{2} > 0$

$f'\left(\frac{5\pi}{4}\right) = 0$ and $f''\left(\frac{5\pi}{4}\right) > 0$

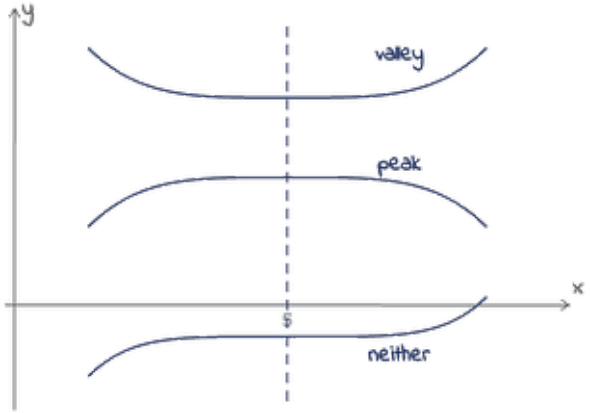
→ rel. min at $x = \frac{5\pi}{4}$

Workbook → $f' = 0$, sign of f'', both facts

PROBLEM 2 OF 3

Which must be true?

f is twice differentiable. $f'(2) = 0$, $f''(2) = -3$, $f'(5) = f''(5) = 0$. Which must be true?
(I) f has a relative maximum at $x = 2$.
(II) f has a relative minimum at $x = 5$.
(III) f has a point of inflection at $x = 5$.
(A) I only (B) II only (C) I and III only (D) I, II, and III



Solution One statement at a time

Plan: I, II, III

(I) $f'(2) = 0$, $f''(2) = -3 \rightarrow$ rel. max ✓

(II) $f''(5) = 0 \rightarrow$ no verdict

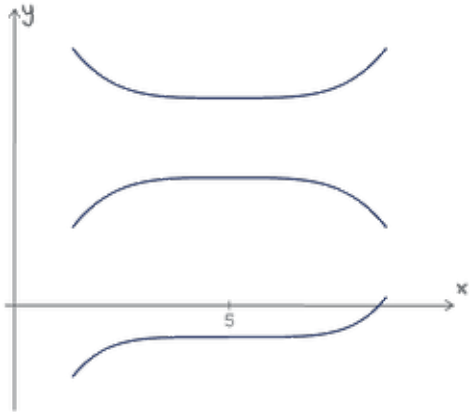
Peak possible → not guaranteed ✗

(III) cup or cap on both sides → not guaranteed ✗

→ (A)

Workbook → one statement at a time

when $f''(5) = 0$



✗ Common mistake

$f''(5) = 0 \rightarrow$ inflection?

$f''(5) = 0 \rightarrow$ no extremum?

✓ Do this

$f''(5) = 0 \rightarrow$ no verdict

Extremum: f' changes sign

Inflection: f'' changes sign

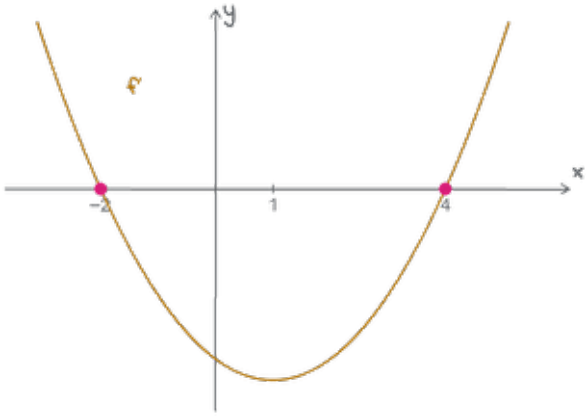
PROBLEM 3 OF 3

Given f', relative and absolute

f is differentiable and $f'(x) = x^2 - 2x - 8$.

(a) Where does f have relative extrema? Use the Second Derivative Test.

(b) On $x > -2$, does f have an absolute minimum ? Justify your answers.



Solution (a) Relative extrema

(a)

Plan → $f' = 0$, then sign of f''

Given f' (not f)

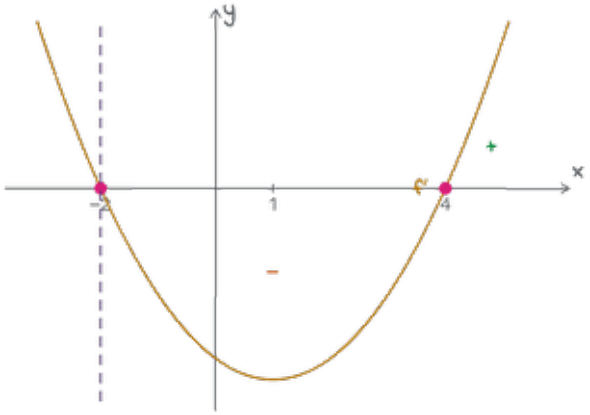
$x^2 - 2x - 8 = (x-4)(x+2) = 0 \rightarrow x = -2, 4$

$f'' = 2x - 2$

$f''(-2) = -6 < 0 \rightarrow$ rel. max at $x = -2$

$f''(4) = 6 > 0 \rightarrow$ rel. min at $x = 4$

Solution (b) Only one critical point



(b)

$x > -2$ → boundary at -2

-2 left out → only critical point → $x = 4$

differentiable → continuous on $x > -2$

f' → − on $(-2, 4)$, + after 4

→ yes: absolute min at $x = 4$

Only critical point + f' goes − → +

Workbook → relative to absolute

WRAP-UP

Peak or valley in one number?

Cup → min · cap → max

$f''(c) = 0$ → back to signs of f'

Only one critical point → absolute

Both → $f'(c) = 0$ and the sign of $f''(c)$

Next → 5.8: sketch f and f'

Next: the workbook for this topic (link in the video description). Stuck? Ask in the comments.