5.7 / The Second Derivative Test

Watch the lesson ➤

Using the second derivative test to determine extrema

Learning goal

Use $f'(c) = 0$ and the sign of $f''(c)$ to classify a critical point, know when the test cannot tell, and use one critical point to find an absolute extremum.

Key ideas

01 $f(x) = x^3 - 12x$: $f'(x) = 3x^2 - 12$, $f''(x) = 6x$. At 2, $f''(2) = 12 > 0$: f' rises through 0, − to +, a **relative min**. At -2, $f''(-2) = -12 < 0$: a **relative max**.

02 The **Second Derivative Test**: $f'(c) = 0$ and $f''(c) > 0$: relative min. $f'(c) = 0$ and $f''(c) < 0$: relative max. Only where $f'(c) = 0$.

03 When $f''(c) = 0$ the test **can't tell**: x^4 (valley), $-x^4$ (peak), x^5 (neither) all have $y'(0) = y''(0) = 0$. Use the signs of f'.

04 Only one critical point: if f is continuous on an interval with only one critical point c, and f has a relative extremum at c, it is absolute there. $f(x) = x + \frac{4}{x}$, $x > 0$: absolute min 4 at $x = 2$.

Remember

Second Derivative Test
$f'(c) = 0$: $f''(c) > 0$ gives a relative min, $f''(c) < 0$ a relative max, $f''(c) = 0$ no conclusion.

twice differentiable
f' and f'' exist.

Class notes

5.7 / Write it clearly

What earns the point on the AP Exam, and the mistakes that lose it.

Common mistakes

The mistake code is used in the answers and in My Mistakes

01 f” = 0 is not an answer

✗ $f''(5) = 0$, so f has a point of inflection at 5 (or: no relative max or min at 5).

✓ $f''(c) = 0$: no conclusion yet. Check the sign changes of f' and f''.

5.7 / Guided examples

Watch, pause, and write the reasoning. Full solutions are in the video.

Problem types

Read the problem, name the type, then use its method

Type	Method	Problems
The Second Derivative Test	$f' = 0 \to$ the sign of f'' at each point $\to$ relative min or max.	E01 · Practice 1, 2, 4
When f" $= 0$	$f''(c) = 0$: the test can't tell; use the signs of f'.	E02 · Practice 3, 6
Only one critical point	Continuous on an interval, one critical point, a relative extremum there $\to$ absolute.	E03 · Practice 5

Example 01

Video · E01

Let $f(x) = \sin x + \cos x$ on $(0, 2\pi)$.
Use the Second Derivative Test to find where f has a relative maximum and a relative minimum. Justify.

> Find $f' = 0$ first. Then only the sign of f'' there.

Example 02

Video · E02

f is twice differentiable. $f'(2) = 0$, $f''(2) = -3$, $f'(5) = f''(5) = 0$.
Which must be true?
(I) f has a relative maximum at $x = 2$.
(II) f has a relative minimum at $x = 5$.
(III) f has a point of inflection at $x = 5$.

A. I only
B. II only
C. I and III only
D. I, II, and III

> One statement at a time. $f''(c) = 0$ gives no conclusion.

Example 03

Video · E03

f is differentiable and $f'(x) = x^2 - 2x - 8$.
(a) Where does f have relative extrema? Use the Second Derivative Test.
(b) On $x > -2$, does f have an absolute minimum? Justify.

> On $x > -2$ there is only one critical point.

VIDEO COMPLETE Check your example answers at the end of this booklet.

5.7 / Example answers

The Second Derivative Test

E01 $f'(x) = \cos x - \sin x = 0 \;\rightarrow\; \tan x = 1 \;\rightarrow\; x = \frac{\pi}{4},\ \frac{5\pi}{4}$

$f''(x) = -\sin x - \cos x, \qquad f''\left(\frac{\pi}{4}\right) = -\sqrt{2} < 0, \qquad f''\left(\frac{5\pi}{4}\right) = \sqrt{2} > 0$

$f' = 0$ and $f'' < 0$ at $\frac{\pi}{4}$: a relative maximum. $f' = 0$ and $f'' > 0$ at $\frac{5\pi}{4}$: a relative minimum.

E02 (A) (I) $f'(2) = 0$ and $f''(2) = -3 < 0$: a relative maximum. True.
(II), (III) $f''(5) = 0$: the test cannot tell. Like x^4, $-x^4$, x^5 at 0, it can be a valley, a peak, or neither.
Only I must be true. Answer A.

E03 $f'(x) = (x-4)(x+2) = 0 \;\rightarrow\; x = -2,\ 4, \qquad f''(x) = 2x - 2$
(a) $f''(-2) = -6 < 0$: relative max at $x = -2$. $f''(4) = 6 > 0$: relative min at $x = 4$.
(b) f is differentiable, so continuous. On $x > -2$ the only critical point is $x = 4$, a relative min, so it is the absolute minimum. Yes.