Lecture Notes · Unit 5 · Lecture 6 AP CED 5.6

Concavity and points of inflection

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INTRO: TWO TRAILS, BOTH UPHILL

Concavity and Points of Inflection AP Calculus 5.6

Both uphill → $f' > 0$

Gentle → steep | Steep → gentle

Both $f' > 0$ → tell their bends apart?

Bend → flip point → justify · f' graph

BIG IDEA 1: WATCH THE SLOPES

watch the slopes

Green: 0.5, 1, 1.5

Slopes grow → f' increasing

→ like a cup: concave up

Orange: 1.5, 1, 0.5

Slopes shrink → f' decreasing

→ like a cap: concave down

CUP OR CAP: THE SIGN OF F''

Cup or cap: f''

Cup: tangent below · cap: tangent above

Check f' increasing → its derivative: f''

5.3: $f'' > 0 \rightarrow f'$ increasing → f concave up

$f'' < 0 \rightarrow f'$ decreasing → f concave down

$y = ax^2 + bx + c \rightarrow y' = 2ax + b \rightarrow y'' = 2a$

$a > 0$ → cup · $a < 0$ → cap

BIG IDEA 2: WHERE THE BEND FLIPS

where the bend flips

Left: bends down (cap)

Right: bends up (cup)

At 0 → the bend flips

Concavity changes → point of inflection

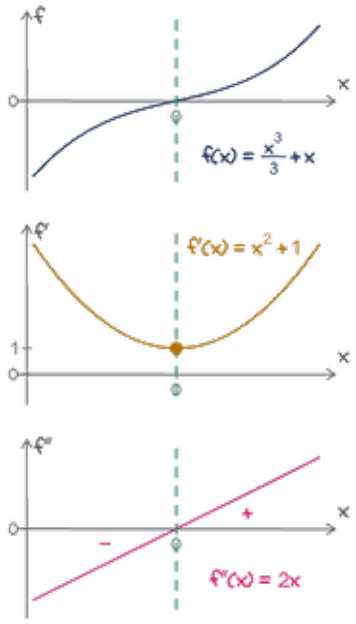
Across it → f'' changes sign

5.4: $f' = 0$ · $f'' = 0$ → only a candidate

Check: does f'' change sign?

F, F', F'' ON ONE X

f, f', f'' on one x



$f' = x^2 + 1 > 0 \rightarrow f$ increases

$x < 0$: $f'' < 0 \rightarrow f'$ falls → cap

$x > 0$: $f'' > 0 \to f'$ rises $\to$ cup

At 0: f'' − → + → inflection at $(0, 0)$

f' ↓ then ↑ at 0: lowest point

f' max or min → f: inflection

BIG IDEA 3: THE SENTENCES TO WRITE

The sentences

Formula: $f''(x) > 0$ on $(a, b) \to f$ concave up

Graph: f' increasing on $(a, b) \to f$ concave up

Formula: f'' changes − → + at $x = c \to$ inflection

Graph: f' falls then rises at $x = c \to$ inflection

f' valley or peak → inflection of f

Name the function, never "it" · $f'' = 0$ alone → not a reason

CONCAVITY AT ONE POINT FROM DY/DX

Concavity at one point

$\frac{dy}{dx} = x^2 - y$ bend at $(1, 2)$?

Differentiate again

$$\frac{d^2y}{dx^2} = 2x - \frac{dy}{dx}$$

At $(1, 2)$: $\frac{dy}{dx} = 1^2 - 2 = -1$

$$\frac{d^2y}{dx^2} = 2(1) - (-1) = 3 > 0$$

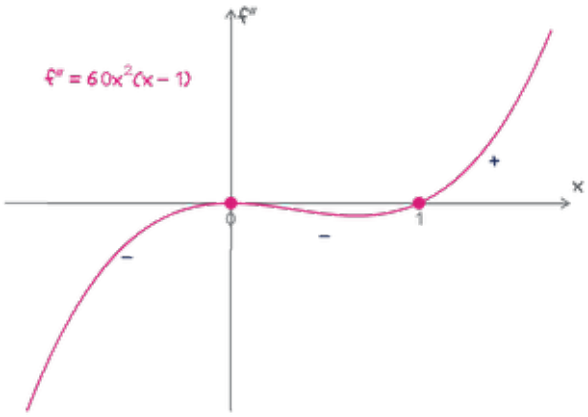
At $(1, 2)$, $y'' > 0$: concave up

PROBLEM 1 OF 3

Concavity and the tempting zero

Let $f(x) = 3x^5 - 5x^4$.

Find the intervals where f is concave up and concave down, and all x where f has a point of inflection. Justify your answer.



Solution Find f'' and read its sign

Plan: $f'' \to$ its zeros $\to$ its sign $\to$ sentences

$f'(x) = 15x^4 - 20x^3$

$f''(x) = 60x^3 - 60x^2 = 60x^2(x - 1)$

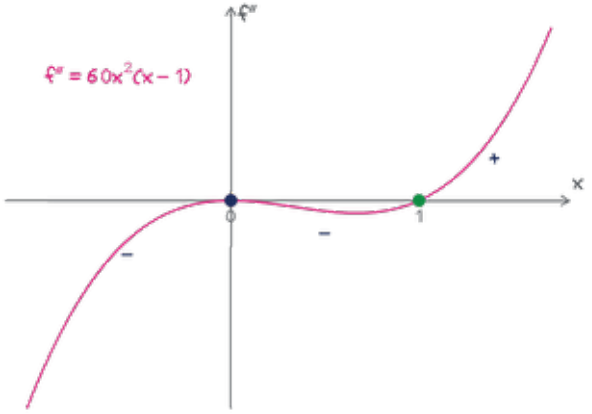
$f'' = 0 \to x = 0,\ x = 1 \to$ candidates

Sketch f'': cross at 1, touch at 0

Sign of f'': −, −, +

Your turn → the sentence for $x = 1$

Solution Intervals and points of inflection



$f'' < 0$ on $(-\infty, 0)$ and $(0, 1)$ → concave down

$f'' > 0$ on $(1, \infty)$ → concave up

$x = 1$: f'' − → + → inflection

f'': − → + → f' falls then rises → min at 1

$f''(0) = 0$: − → − → no inflection. f' keeps falling

Bonus: $f' = 5x^3(3x - 4)$, + → − at 0 → relative max

Workbook → test signs at every zero

COMMON MISTAKE: F'' = 0 ALONE

Common mistake

✗ "$f''(0) = 0$, so f has a point of inflection at 0."

✓ Same sign on both sides → not an inflection

Reason → sign change

PROBLEM 2 OF 3

Where does it flip?

Let $f(x) = \ln(x^2 + 4)$. At what values of x does f have a point of inflection?

(A) 0 only

(B) ±1

(C) ±2

(D) ±4

Solution f' and f''

Plan: f', f'' → zeros of f'' → sign

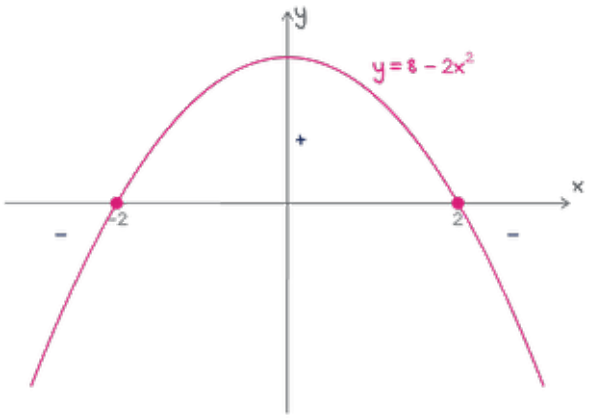
$$(\ln u)' = \frac{u'}{u}, \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$f'(x) = \frac{2x}{x^2 + 4}$$

$$f''(x) = \frac{2(x^2 + 4) - 2x \cdot 2x}{(x^2 + 4)^2}$$

Your turn → simplify the top

Solution Read the sign of the top



Top: $2x^2 + 8 - 4x^2 = 8 - 2x^2$

Bottom $(x^2 + 4)^2 > 0$ → sign of f'' = sign of $8 - 2x^2$

$8 - 2x^2 = 0 \to x^2 = 4 \to x = \pm 2$

$y = 8 - 2x^2$: − outside, + inside

−2, 2: sign changes → (C)

Trap (A): 0 is where $f' = 0$, a critical point

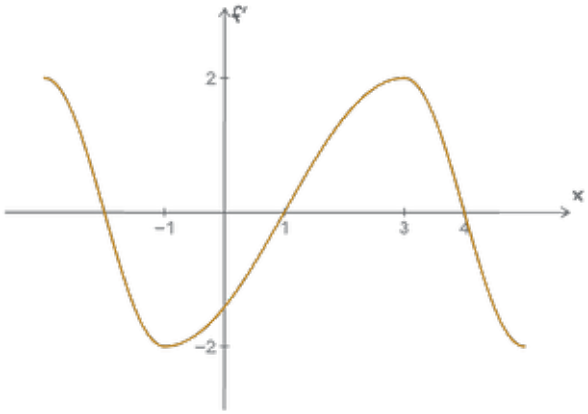
workbook → bottom > 0 → read the top

PROBLEM 3 OF 3

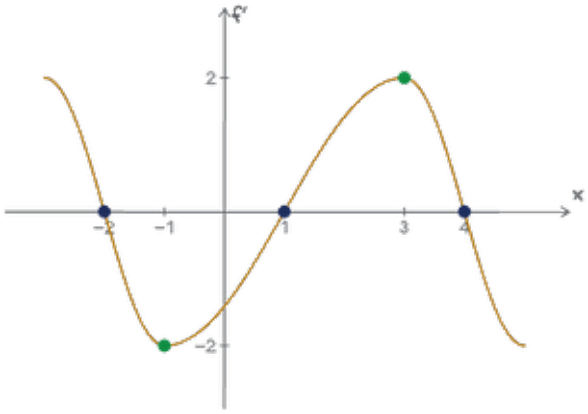
Concave up from the graph of f'

The graph of f', the derivative of f, is shown for $-3 \le x \le 5$. On which interval is the graph of f concave up?

(A) $(-3, -1)$
(B) $(-1, 3)$
(C) $(1, 4)$
(D) $(3, 5)$



Solution Rising or falling, not above or below



This graph is f' → concave up ⇔ f' rising

(A) (−3, −1): f'↓ → ×

(−1, 3): f' rises → (B)

(D) (3, 5): f'↓ → ×

(C) (1, 4): f' > 0 → f increasing, not the bend

So (B) (−1, 3)

−1, 3: f' turns → f changes concavity

Zeros of f': −2, 1, 4 → critical points, not inflections

workbook → is f' rising or falling?

WRAP-UP

Two uphill bends

watch f': rising → cup, falling → cap

Algebra → the sign of f''

Bend flips ⇔ f' has a max or min

f'' = 0 → a candidate → prove the sign change

Peak: cap · valley: cup → 5.7 Second Derivative Test

Next: the workbook for this topic (link in the video description). Stuck? Ask in the comments.