5.1 / Using the Mean Value Theorem

Watch the lesson ➤

Rolle's Theorem and the Mean Value Theorem

Learning goal

Check the conditions, then use Rolle's Theorem or the Mean Value Theorem to say a slope must happen, and pick the right theorem: a value of f' or a value of f.

Key ideas

01 Rolle's Theorem: (i) f continuous on $[a, b]$, (ii) differentiable on (a, b), (iii) $f(a) = f(b)$. Then there is **at least one** c in (a, b) with $f'(c) = 0$.

02 Why both conditions: $|x|$ on $[-1, 1]$ has a corner and no flat tangent (not differentiable); a jump at an end never has to be flat (not continuous).

03 Tilt it, the Mean Value Theorem: (i) and (ii) only. Then at least one c in (a, b) has $f'(c) = \dfrac{f(b) - f(a)}{b - a}$: a tangent slope equals the secant slope. Rolle is the case with average 0.

04 What to write: *f is differentiable, so it is continuous on $[a, b]$ and differentiable on (a, b).* Want f' equal to the average: the MVT. Equal ends and $f' = 0$: Rolle. Want a value of f: the IVT. A condition fails: no promise.

Remember

Rolle's Theorem

Continuous on $[a, b]$, differentiable on (a, b), $f(a) = f(b)$: some c in (a, b) has $f'(c) = 0$.

Mean Value Theorem

Continuous on $[a, b]$, differentiable on (a, b): some c in (a, b) has $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Class notes

5.1 / Write it clearly

What earns the point on the AP Exam, and the mistakes that lose it.

Common mistakes

The mistake code is used in the answers and in My Mistakes

01 The wrong theorem

✗ W is continuous, so by the IVT, $W'(t) = 4$.

✓ Want W': the MVT. The IVT gives a value of W, not W'. And the reason first: W is differentiable, so it is continuous.

5.1 / Guided examples

Watch, pause, and write the reasoning. Full solutions are in the video.

Problem types

Read the problem, name the type, then use its method

Type	Method	Problems
Find the c of the Mean Value Theorem	Conditions, then the average rate of change, then solve $f'(c) =$ that average; keep c inside.	E01 · Practice 1, 3
Rolle's Theorem	Equal end values: $f'(c) = 0$ somewhere inside.	Practice 2
Which theorem, from a table	A rate -> MVT (Rolle if the ends match); a value -> IVT. Differentiable, so continuous.	E02 · Practice 4
When the conditions fail	Check both conditions before using the theorem.	E03 · Practice 5, 6, 7

Example 01

Video · E01

Let $f(x) = x^3 - 2x$ on the interval $[0, 3]$.
(a) Show f is continuous on $[0, 3]$ and differentiable on $(0, 3)$.
(b) Find all c in $(0, 3)$ that satisfy the conclusion of the MVT.

> Check both conditions first. Then set $f'(c)$ equal to the average rate of change.

Example 02

Video · E02

Water temperature $W(t)$ in °C at t minutes. W is differentiable. Justify each.
(a) Must $W'(t) = 4$ for some t in $(0, 3)$?
(b) Must $W'(t) = 0$ for some t in $(3, 12)$?
(c) Must $W(t) = 40$ for some t in $(0, 7)$?

t	0	3	7	12
$W(t)$	20	32	44	32

A value of W': the MVT (or Rolle). A value of W: the IVT.

Example 03

Video · E03

Which function satisfies the hypotheses of the Mean Value Theorem on the given interval?

A. $f(x) = |x - 2|$ on $[0, 4]$
B. $f(x) = \dfrac{1}{x - 1}$ on $[0, 2]$
C. $f(x) = x^{\frac{2}{3}}$ on $[-1, 1]$
D. $f(x) = \sqrt{x}$ on $[0, 4]$

Continuous on the closed interval, differentiable on the open one.

VIDEO COMPLETE Check your example answers at the end of this booklet.

5.1 / Example answers

Using the Mean Value Theorem

E01 f is a polynomial, so it is continuous on $[0, 3]$ and differentiable on $(0, 3)$.
$\frac{f(3)-f(0)}{3-0} = \frac{21-0}{3} = 7$
$f'(c) = 3c^2 - 2 = 7 \;\rightarrow\; c^2 = 3 \;\rightarrow\; c = \pm\sqrt{3}$
$-\sqrt{3}$ is not in $(0, 3)$, so $c = \sqrt{3}$.

E02 W is differentiable, so it is continuous on every interval.
(a) $\frac{W(3)-W(0)}{3-0} = \frac{32-20}{3} = 4$. By the MVT, yes.
(b) $W(3) = W(12) = 32$. By Rolle's Theorem, yes: $W'(t) = 0$ for some t in $(3, 12)$.
(c) $W(0) = 20 < 40 < 44 = W(7)$. By the IVT, yes.

E03 (D) (A) has a corner at 2, (B) breaks at 1: both inside the interval.
(C) $f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$ does not exist at 0, inside $(-1, 1)$.
(D) $\sqrt{x}$ is continuous on $[0, 4]$; $f'(x) = \frac{1}{2\sqrt{x}}$ fails only at the endpoint 0. Answer D.