Lecture Notes · Unit 4 · Lecture 9 AP CED 4.7

L'Hospital's Rule

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INTRO: A 0/0 THAT ALGEBRA CANNOT CRACK

L'Hospital's Rule AP Calculus 4.7

$\lim_{x \to 0} \frac{\sin 3x}{e^x - 1} = ?$

Top: $\sin 0 = 0$

Bottom → $\lim_{x \to 0} (e^x - 1) = 0$

Factor? Conjugate? → nothing

Not a value → not decided yet

Today → one line, and when NOT to

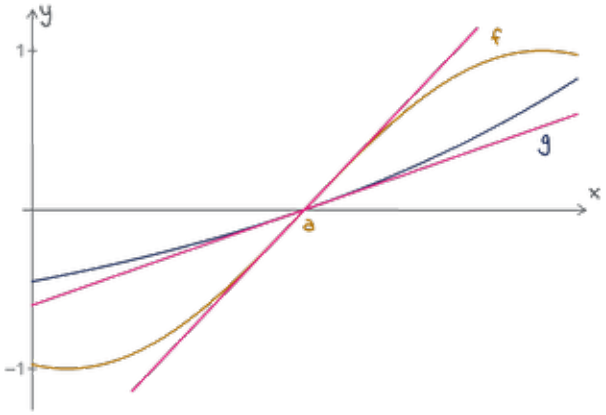
WHY IT WORKS: ZOOM IN ON TWO CURVES

Why: two tangent lines

Up close → curve ≈ tangent line

$L(x) = f(a) + f'(a)(x - a)$

An idea, not a proof (proof → college)



2 curves → $f(a) = 0$, $g(a) = 0$

$f(x) \approx f'(a)(x - a)$

$g(x) \approx g'(a)(x - a)$

$\frac{f(x)}{g(x)} \approx \frac{f'(a)(x - a)}{g'(a)(x - a)}$

$x - a$ divides out → $\frac{f'(a)}{g'(a)}$

Two lines through $(a, 0)$ → ratio of slopes

(this picture → $g'(a) \neq 0$)

THE RULE: CHECK FIRST

L'Hospital's Rule

f, g differentiable near a; $g' \neq 0$ near a

Check first → top → 0 and bottom → 0

or top → $\pm\infty$ and bottom → $\pm\infty$

Name → $\frac{0}{0}, \frac{\infty}{\infty}$ = indeterminate forms

Then → slope of top ÷ slope of bottom

$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

$\frac{0}{0}$ = a label, not a value → never write "$= \frac{0}{0}$"

Name → L'Hospital's Rule

Why $\frac{\infty}{\infty}$ too? → same deal an idea, not a proof

$f, g \to \infty$ → flip → $\frac{1}{f} \to 0, \frac{1}{g} \to 0$

$\frac{f}{g} = \frac{\frac{1}{g}}{\frac{1}{f}}$ → a $\frac{0}{0}$ form

Derivatives → $\frac{-\frac{g'}{g^2}}{-\frac{f'}{f^2}}$

$= \frac{g'}{f'}\left(\frac{f}{g}\right)^2$

$\lim_{x\to a}\frac{f}{g}=L\ (L\neq 0)\to L=\lim_{x\to a}\frac{g'}{f'}\cdot L^2$

$1=L\lim_{x\to a}\frac{g'}{f'}\Rightarrow\frac{1}{L}=\lim_{x\to a}\frac{g'}{f'}\to L=\lim_{x\to a}\frac{f'}{g'}$

$\lim_{x\to 0}\sin 3x=0,\ \lim_{x\to 0}(e^x-1)=0$

$\lim_{x\to 0}\frac{\sin 3x}{e^x-1}=\lim_{x\to 0}\frac{3\cos 3x}{e^x}$

$=\frac{3\cos 0}{e^0}=3\to$ done

OUR LIMIT, AND WHEN NOT TO USE THE RULE

Before you use it: two warnings

warning 1 → $\lim_{x\to 0}\frac{x+1}{x+2}$ → top → 1, bottom → 2

$=\frac{1}{2}\to$ never $\frac{0}{0}$

Rule anyway → $\frac{1}{1}=1$ X

top → 3, bottom → 0 → not $\frac{0}{0}$ either

warning 2 → $\frac{f'}{g'}$, not $\left(\frac{f}{g}\right)'$

Quotient rule $\frac{f'g-fg'}{g^2}$ → not this

other forms ($\infty-\infty$, ...) → not on the AP exam

UNIT 1 LIMITS IN ONE LINE; WHO GROWS FASTER

Old limits, one line

$\lim_{x\to 2}\frac{x^2+2x-8}{x-2}$

top → $4+4-8=0$, bottom → 0

$=\lim_{x\to 2}\frac{2x+2}{1}=6$

$\lim_{x\to\infty}\frac{\ln x}{x}\to$ both → ∞

$=\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=0$

→ x grows much faster than ln x

grows much slower than → < <

$n>0,\ x\to\infty\to \ln x<<x^n<<e^x$

$\lim_{x\to\infty}\frac{x}{e^x}=\lim_{x\to\infty}\frac{1}{e^x}=0$

PROBLEM 1 OF 3

A limit from a table

f has a continuous derivative. Selected values are in the table.

(a) Find $\lim_{x\to 1}\frac{f(x)-4}{x^2-1}$. Show that L'Hospital's rule applies.

(b) Find $\lim_{x\to 1}\frac{f(x)}{x+1}$.

x	f(x)	f'(x)
1	4	6

Solution (a) Check, then the rule

(a)

Plan: at $x=1$ → top → 0? bottom → 0? → then the rule

No formula for f → two values are enough

f differentiable → continuous at 1

$\lim_{x\to 1}(f(x)-4)=f(1)-4=4-4=0$

$\lim_{x\to 1}(x^2-1)=1-1=0$

Both → 0; $(x^2-1)'=2x\neq 0$ near 1 → rule applies

Your turn → the next line

$\lim_{x\to 1}\frac{f(x)-4}{x^2-1}=\lim_{x\to 1}\frac{f'(x)}{2x}$

$=\frac{f'(1)}{2(1)}=\frac{6}{2}=3$

Solution (b) Same table: check the form first

x	$f(x)$	$f'(x)$
1	4	6

(a) both → 0 → rule → 3

(b)

Plan: check form → rule

$\lim_{x \to 1} f(x) = f(1) = 4$

$\lim_{x \to 1} (x+1) = 2$

$\frac{4}{2}$ → not $\frac{0}{0}$ → no rule

$\lim_{x \to 1} \frac{f(x)}{x+1} = \frac{4}{2} = 2$

Rule anyway → $\frac{f'(1)}{1} = 6$ X

Same table → the first line decides

Workbook → form first

COMMON MISTAKE: = 0/0 AND A MISSING LIM

One student's answer

X $\lim_{x \to 1} \frac{f(x)-4}{x^2-1} = \frac{0}{0}$

$\frac{0}{0}$ is not a number → false equation

$X = \frac{f'(x)}{2x} = 3$

No lim → says $\frac{f'(x)}{2x} = 3$ for every x

✓ top → 0, bottom → 0, written separately

✓ keep $\lim_{x \to 1}$ until you plug in

PROBLEM 2 OF 3

The rule, twice

$\lim_{x \to 0} \frac{e^{2x} - 1 - 2x}{x^2} =$

(A) 0

(B) 1

(C) 2

(D) nonexistent

$$\frac{d}{dx} e^{2x} = 2e^{2x}$$

Solution Check, rule, check again

Plan: check → rule → check again

top → 1 − 1 − 0 = 0, bottom → 0

$$\lim_{x \to 0} \frac{e^{2x} - 1 - 2x}{x^2} = \lim_{x \to 0} \frac{2e^{2x} - 2}{2x}$$

Again → top → 2 − 2 = 0, bottom → 0 → still $\frac{0}{0}$

$$= \lim_{x \to 0} \frac{4e^{2x}}{2} = \frac{4}{2} = 2$$

Not $\frac{0}{0}$ anymore → stop and plug in

So (C)

Workbook → 3 rounds → check each

PROBLEM 3 OF 3

Infinity over infinity

Find $\lim_{x \to \infty} \frac{\ln(x^2+1)}{\ln x}$.

$$\frac{d}{dx} \ln u = \frac{u'}{u}$$

Solution Both to infinity, then the rule

Plan: both → ∞? → then the rule

$\ln(x^2+1) \to \infty$, $\ln x \to \infty$

$$\lim_{x\to\infty}\frac{\ln\left(x^2+1\right)}{\ln x}=\lim_{x\to\infty}\frac{\frac{2x}{x^2+1}}{\frac{1}{x}}$$

$$=\lim_{x\to\infty}\frac{2x^2}{x^2+1}$$

$$=\lim_{x\to\infty}\frac{2}{1+\frac{1}{x^2}}$$

$$\frac{1}{x^2}\to 0\to\frac{2}{1+0}=2$$

Workbook → another infinity over infinity

WRAP-UP

Check first, then the rule

$\frac{0}{0}$ → a label

$\frac{0}{0}$ or $\frac{\infty}{\infty}$ → slope of top ÷ slope of bottom

Neither $\frac{0}{0}$ nor $\frac{\infty}{\infty}$ → don't use the rule

Unit 4 ✓ → Next: 5.1

Next: the workbook for this topic (link in the video description). Stuck? Ask in the comments.