Lecture Notes · Unit 4 · Lecture 8 AP CED 4.6

Local linearity and linearization

Tap a blue play button next to a heading to open that part of the video. ▶ Watch the video

INTRO: THE SQUARE ROOT OF 4.1, NO CALCULATOR

Linear Approximation AP Calculus 4.6

$\sqrt{4.1}$ = ? → no calculator

First guess 2.1 → $2.1^2 = 4.41 > 4.1$

$2^2 = 4 \rightarrow 2 < \sqrt{4.1} < 2.1$

How close? → one tangent line

Estimate → too high or too low?

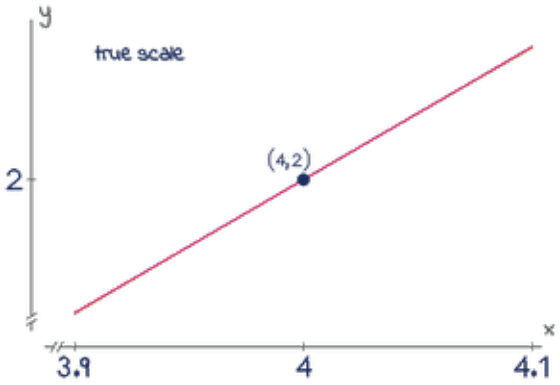
ZOOM IN: A CURVE BECOMES A LINE

Up close, a curve is a line

Tangent at (4, 2)

At 4 → same height, same slope

Near 4 → almost together



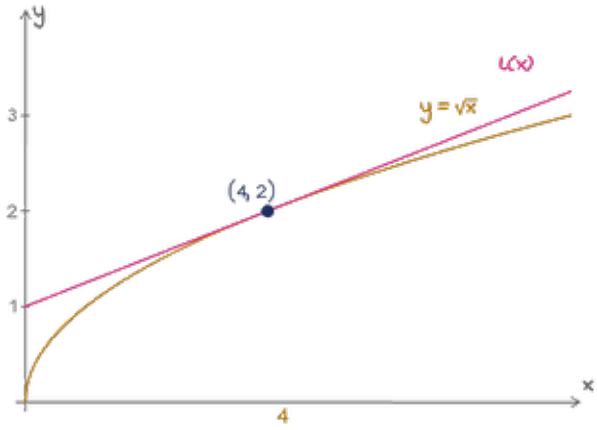
Zoom more → can't tell them apart

Yes → near 4, the line stands in

Name → local linearity

BUILD THE TANGENT LINE, THEN PLUG IN

The tangent line does the work



Height = height at a + slope × (x − a)

$L(x) = f(a) + f'(a)(x - a)$

Point-slope form (2.2) → nothing new

a = 4 → $\sqrt{4}$ is clean, next to 4.1

$f(4) = \sqrt{4} = 2$

$f'(x) = \frac{1}{2\sqrt{x}} \rightarrow f'(4) = \frac{1}{2 \cdot 2} = \frac{1}{4}$

$L(x) = 2 + \frac{1}{4}(x - 4)$

$L(4.1) = 2 + \frac{1}{4}(4.1 - 4)$

$= 2 + \frac{1}{4}(0.1) = 2.025$

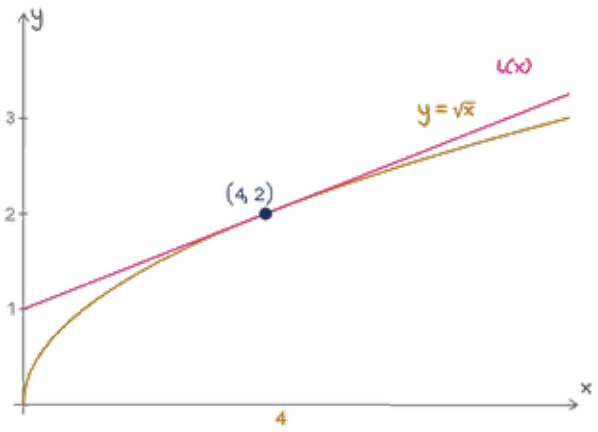
about → $\approx \sqrt{4.1} \approx 2.025$

Check → $2.025^2 = 4.100625$

Name → local linear approximation

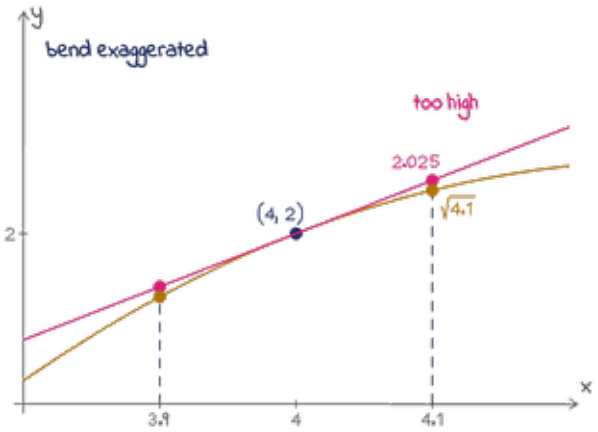
TOO HIGH OR TOO LOW?

Over or under?



$\sqrt{4.1} \approx 2.025$ (tangent at 4)

Over or under?



At 4.1 → line above curve

Curve bends down; line goes straight

Line above → 2.025 is an overestimate

$2.025^2 = 4.100625 > 4.1$

Left of 4 → still below the line

NO PICTURE? THE SIGN OF F''

Which way does it bend?



f'' = how the slope changes

f'' < 0 → the slope shrinks

→ bends down, below its tangents → over

Concave down → Unit 5

f'' > 0 → slope grows

f bends up → above tangent → under

Trap 1 → not the sign of f or f'

Trap 2 → the sign on the whole interval

Which graph bends? → f

PROBLEM 1 OF 3

Estimate, then over or under

A function f has $f(1) = 4$ and $f'(x) = \sqrt{x^3 + 8}$.

(a) Use the tangent line at $x = 1$ to approximate $f(1.2)$.

(b) Is it an underestimate or an overestimate? Explain your reasoning.

Tangent line →

$L(x) = f(a) + f'(a)(x - a)$

Solution (a) Point, slope, line

(a)

Plan: (1, f(1)), slope → L(x) → L(1.2)

No formula for f → a point + a slope is enough

f(1) = 4

$f'(1) = \sqrt{1^3 + 8} = \sqrt{9} = 3$

$L(x) = 4 + 3(x - 1)$

$f(1.2) \approx 4 + 3(1.2 - 1)$

$= 4 + 3(0.2) = 4.6$

Your turn → (b): over or under?

Solution (b) The sign of f' on [1, 1.2]

Chain rule →

$\frac{d}{dx}\sqrt{u} = \frac{u'}{2\sqrt{u}}$

(b)

Plan: sign of f'' on [1, 1.2]

$f''(x) = \frac{3x^2}{2\sqrt{x^3 + 8}}$

Top $3x^2 > 0$, bottom > 0

$f''(x) > 0$ for every x in [1, 1.2]

f bends up → above tangent

Since $f'' > 0$ on [1, 1.2] → 4.6 is an underestimate

Workbook → estimate, classify

COMMON MISTAKE: ONE POINT IS NOT ENOUGH

One point is not enough

✗ $f''(1) > 0$ → underestimate

1 point → 1 snapshot

[1, 1.2]: same bend?

✓ $f'' > 0$ on [1, 1.2]

f above tangent → under

PROBLEM 2 OF 3

A cube root without a calculator

The tangent line to $y = \sqrt[3]{x}$ at $x = 27$ is used to approximate $\sqrt[3]{26.7}$. Which of the following is true about the approximation?

(A) $\frac{269}{90}$, an overestimate

(B) $\frac{269}{90}$, an underestimate

(C) $\frac{271}{90}$, an overestimate

(D) $\frac{271}{90}$, an underestimate

$\frac{d}{dx}x^n = nx^{n-1}$

$L(x) = f(a) + f'(a)(x - a)$

Solution Point, slope, line, then the bend

Plan: point, slope at 27 → L(x) → sign of f''

$f(27) = \sqrt[3]{27} = 3$

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}}$

$f'(27) = \frac{1}{3(9)} = \frac{1}{27}$

$L(x) = 3 + \frac{1}{27}(x - 27)$

Step left → $26.7 - 27 = -0.3$

$L(26.7) = 3 + \frac{1}{27}(-0.3)$

$= 3 - \frac{1}{90} = \frac{269}{90}$

Solution Point, slope, line, then the bend

$\frac{d}{dx}x^n = nx^{n-1}$

$L(x) = f(a) + f'(a)(x - a)$

Going left → flip? check f″

$f'' = -\frac{2}{9}x^{-\frac{5}{3}} < 0 \quad (x > 0)$

Bends down on both sides → over

So (A)

Workbook root · no calc.

PROBLEM 3 OF 3

A tangent line to a curve

The point $(1, 2)$ is on the curve $x^3 + y^3 = 9$. Use the tangent line at $(1, 2)$ to approximate the y-value on the curve when $x = 1.1$.

$$d(y^3) = 3y^2\,dy$$

Solution The slope, without solving for y

Plan: slope from $\frac{dy}{dx}$ → tangent line

Solve for y? No → a point + a slope

$3x^2\,dx + 3y^2\,dy = 0$

$3x^2 + 3y^2\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{x^2}{y^2}$

$\left.\frac{dy}{dx}\right|_{(1,2)} = -\frac{1^2}{2^2} = -\frac{1}{4}$

Your turn → tangent (1,2)

Solution The line, then x = 1.1

$y = y_1 + m(x - x_1)$

Point (1, 2), slope $-\frac{1}{4}$

$y = 2 - \frac{1}{4}(x - 1)$

$y \approx 2 - \frac{1}{4}(1.1 - 1)$

$= 2 - \frac{1}{4}(0.1) = 2 - 0.025 = 1.975$

Workbook → slope first, then the line

WRAP-UP

Too high or too low?

Tangent line → $\sqrt{4.1} \approx 2.025$

Bends down → over

f″ on the whole interval

Next → 4.7: zoom in on two curves

Next: the workbook for this topic (link in the video description). Stuck? Ask in the comments.