Lecture Notes · Unit 4 · Lecture 7 AP CED 4.5b

Related rates: similar triangles

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INTRO: ONE THING CAN GO IN EARLY

Related Rates: Similar Triangles AP Calculus 4.5 · d = differential

Before d → which substitution?

Cones, shadows → similar △s

When ratios fail

IDEAS 1–2: LOW VS HIGH, ONE VARIABLE

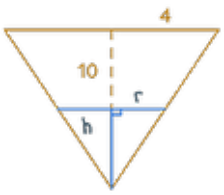
A cone filling up

Cone, point down → in at 2 ft³/min

Level rises at a steady rate?

1. Low, then high

same 2 ft³: low, then high



Low → narrow → a thick layer

High → wide → a thin layer

→ fast at first, then slower

2. Volume of a cone

$V = \frac{1}{3}\pi r^2 h$

r, h both move → d gives dr and dh

Small △, big △ → similar → $\frac{r}{h} = \frac{4}{10}$

$r = \frac{2}{5}h$ → true at every moment

one variable instead of two

IDEAS 3–4: SUBSTITUTE FIRST, OR NOT

Substitute, then take d

3. Substitute for r first

$V = \frac{1}{3}\pi\left(\frac{2h}{5}\right)^2 h$

$= \frac{4\pi}{75}h^3$

$dV = \frac{4\pi}{75} \cdot 3h^2\, dh$

$\frac{dV}{dt} = \frac{4\pi}{25}h^2 \cdot \frac{dh}{dt}$

Always true → before d

one moment → after d

4. When NOT to substitute

$\frac{dr}{dt}$, $\frac{dh}{dt}$ both given

Shape varies: r, h → product rule

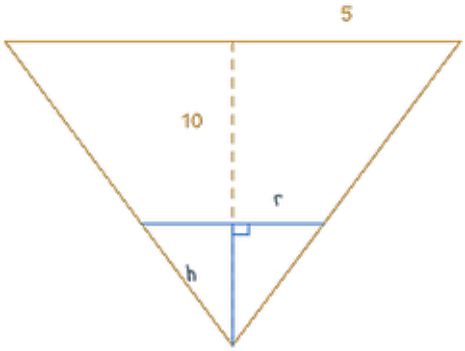
Never invent a ratio

Fixed → number. Moving → variable.

PROBLEM 1 OF 3

Filling a cone

A tank is a cone, point down, with radius 5 ft at the rim and height 10 ft. Water flows in at 2 ft³/min. How fast is the water level rising when the water is 4 ft deep?



Solution Similar triangles, substitute, d, ÷dt

Plan: similar △ → substitute → d → numbers last

$$\frac{r}{h} = \frac{5}{10} \rightarrow r = \frac{h}{2}$$

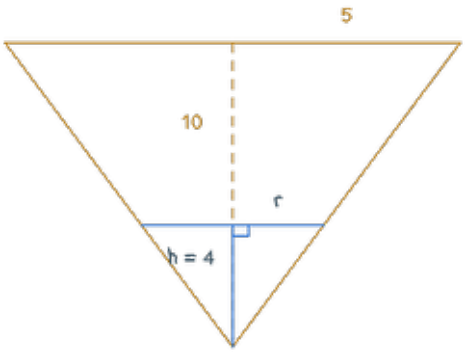
$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

$$dV = \frac{\pi}{12} \cdot 3h^2\, dh$$

$$\div dt \rightarrow \frac{dV}{dt} = \frac{\pi}{4}h^2 \cdot \frac{dh}{dt}$$

Your turn: plug in.

Solution This moment's numbers, last



Plan: $\frac{dV}{dt} = 2$, $h = 4$ go in

$$2 = \frac{\pi}{4}(4)^2\frac{dh}{dt} = \frac{16}{4}\pi\frac{dh}{dt} = 4\pi\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{2}{4\pi} = \frac{1}{2\pi}$$

At $h = 4$ → level rising at $\frac{1}{2\pi}$ ft/min

Workbook → similar triangles first

Careful → r moves with h

X Common mistake

$h = 4 \rightarrow r = 2$ (now) ✓

keep $r = 2 \rightarrow V = \frac{4}{3}\pi h$ X

✓ Do this

$r = \frac{h}{2}$ → r moves with h

Relation → d → values

PROBLEM 2 OF 3

A streetlight and a shadow

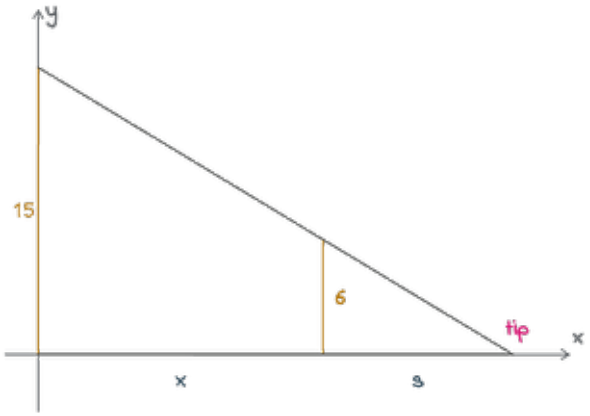
A 6 ft person walks away from a 15 ft streetlight at 5 ft/s. How fast is the tip of the shadow moving?

(A) $\frac{10}{3}$ ft/s

(B) 5 ft/s

(C) $\frac{25}{3}$ ft/s

(D) 2 ft/s



Solution Two triangles, then the tip is x + s

Plan: x = how far you walked, s = shadow length

Small △: 6, s. Big △: 15, x + s

$$\frac{s}{6} = \frac{x+s}{15}$$

$$15s = 6x + 6s \rightarrow 9s = 6x \rightarrow s = \frac{2}{3}x$$

$$ds = \frac{2}{3}dx \rightarrow \frac{ds}{dt} = \frac{2}{3} \cdot 5 = \frac{10}{3}$$

Question: the tip → tip at x + s

$$\frac{d}{dt}(x+s) = 5 + \frac{10}{3} = \frac{25}{3} \text{ ft/s}$$

So (C).

Workbook → length or tip?

PROBLEM 3 OF 3

A melting cone

A block of ice is shaped like a cone. Right now its radius is 6 cm and its height is 10 cm. As it melts, the radius is decreasing at $\frac{1}{2}$ cm/h and the height is decreasing at 1 cm/h. $\left(V = \frac{1}{3}\pi r^2 h\right)$

(a) How fast is the volume changing right now?

(b) Is the volume increasing or decreasing?

Cone

$V = \frac{1}{3}\pi r^2 h$

Product rule

$(uv)' = u'v + uv'$

Melting

rates < 0

Solution Two rates given: test the ratio, keep r and h

(a)

Plan: similar △ allowed? → test first

Melting → $\frac{dr}{dt} = -\frac{1}{2}$, $\frac{dh}{dt} = -1$

If $r = \frac{3}{5}h$ always → $\frac{dr}{dt} = \frac{3}{5}(-1) = -\frac{3}{5}$

Given $-\frac{1}{2} \neq -\frac{3}{5}$ → shape changes → keep r, h

$$dV = \frac{1}{3}\pi\left(2r\,dr \cdot h + r^2\,dh\right)$$

$$\div dt \rightarrow \frac{dV}{dt} = \frac{1}{3}\pi\left(2r\frac{dr}{dt}h + r^2\frac{dh}{dt}\right)$$

Your turn: plug in.

Solution Numbers in, then read the sign

Cone

$V = \frac{1}{3}\pi r^2 h$

Product rule

$(uv)' = u'v + uv'$

Melting

rates < 0

Plan: $r = 6, h = 10$

$$\frac{dv}{dt} = \frac{1}{3}\pi\left(2(6)\left(-\frac{1}{2}\right)(10) + 6^2(-1)\right)$$

$$= \frac{1}{3}\pi(-60 - 36) = -\frac{96}{3}\pi = -32\pi$$

(b)

Negative → volume decreasing at 32π cm^3/h

Workbook: 2 rates → r, h

WRAP-UP

Steady? No

No → thin layers up high

Always true → before d

Two rates given → no ratio

Workbook → video → comments → next: 4.6 local linearity

Next: the workbook for this topic (link in the video description). Stuck? Ask in the comments.