Lecture Notes · Unit 4 · Lecture 6 AP CED 4.5a

Related rates: right triangles and angles

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INTRO: NO EQUATION GIVEN

Related Rates: Right Triangles AP Calculus 4.5

Today no equation

→ build it from a picture

Ladder, boat, two cars, an angle

Fixed → a number. Moving → a variable.

IDEAS 1–2: A SLIDING LADDER

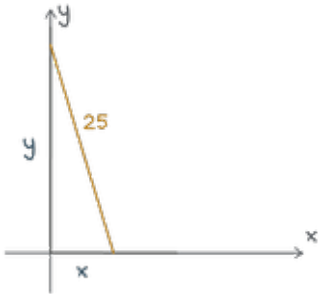
A sliding ladder

Ladder → bottom slides out at 2 ft/s

Top slides down at 2 ft/s too?

1. Try it with a ladder

$x^2 + y^2 = 25^2$



$x = 7 \to 15 \to y = 24 \to 20$

→ out 8, down only 4

$x = 20 \to 24 \to y = 15 \to 7$

→ out 4, down 8 → not the same

2. Fixed or moving?

25 fixed; x, y moving

Ladder doesn't stretch → $d(25^2) = 0$

$2x\,dx + 2y\,dy = 0$

$\div dt \to 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$\frac{dy}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt}$

x grows, y shrinks → $\frac{x}{y}$ grows → speeds up

IDEAS 3–4: SIGNS AND AN ANGLE

Signs; angle

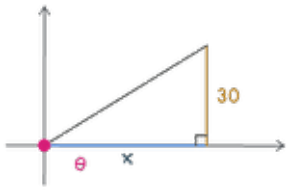
3. Shrinking → negative

y shrinks → $\frac{dy}{dt} < 0$

Closer → negative. Farther → positive.

4. Elevation angle θ

30 m → $\tan\theta = \frac{30}{x}$



$\frac{d}{d\theta}\tan\theta = \sec^2\theta$

$\frac{30}{x} = 30x^{-1}$

$\sec^2\theta\,d\theta = -\frac{30}{x^2}\,dx$

$\div dt \to$

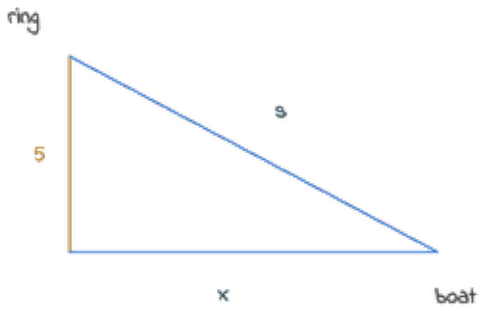
$\sec^2\theta\frac{d\theta}{dt} = -\frac{30}{x^2}\cdot\frac{dx}{dt}$

Don't drop the $\sec^2\theta$

PROBLEM 1 OF 3

A boat and a dock

A boat is pulled toward a dock by a rope. The rope runs from the boat to a ring on the dock, 5 ft above the boat. The rope is pulled in at 2 ft/s. How fast is the boat approaching the dock when 13 ft of rope is out?



Solution Picture, equation, d, ÷dt, numbers last

Plan: sketch → eq. → d → ÷dt → values

5 → fixed → number; x, s → moving → variables

$x^2 + 5^2 = s^2$

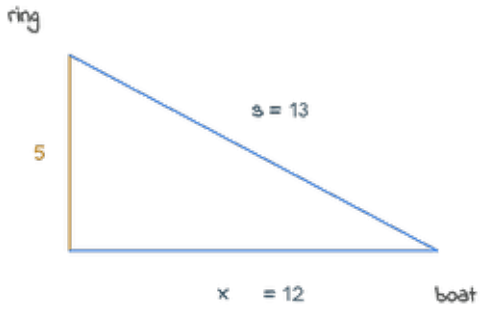
Pulled in → s shrinks → $\frac{ds}{dt} = -2$

$2x\,dx + 0 = 2s\,ds$

$\div dt \rightarrow 2x\frac{dx}{dt} = 2s\frac{ds}{dt}$

Your turn: plug in.

Solution The triangle at this moment



Plan: freeze → x → substitute

$s = 13 \rightarrow x = \sqrt{13^2 - 5^2} = \sqrt{144} = 12$

$2(12)\frac{dx}{dt} = 2(13)(-2)$

$24\frac{dx}{dt} = -52 \rightarrow \frac{dx}{dt} = -\frac{13}{6}$

Negative: x shrinks

Boat approaches the dock at $\frac{13}{6}$ ft/s

$\frac{13}{6} > 2 \rightarrow$ faster than the rope

Workbook → plug in last

Careful → numbers too early

✗ Common mistake

$s = 13 \rightarrow x^2 + 25 = 169$

$d \rightarrow 2x\,dx = 0 \rightarrow \frac{dx}{dt} = 0$?!

✓ Do this

13 at one instant

Fixed 5 first; d → values

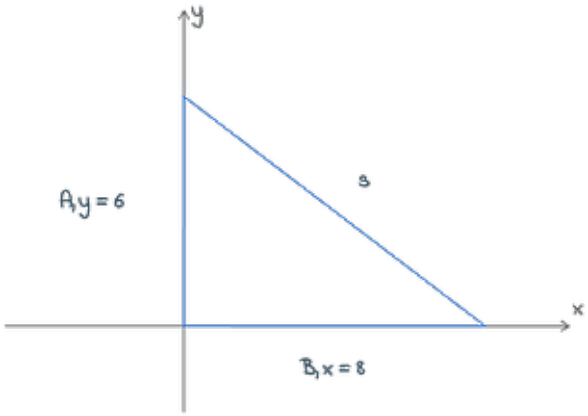
PROBLEM 2 OF 3

Two cars

Car A is 6 mi north of an intersection, driving toward it at 30 mph. Car B is 8 mi east of it, driving away at 40 mph.

(a) How fast is the distance between the cars changing?

(b) Is the distance increasing or decreasing?



Solution Signs first, then $s^2 = x^2 + y^2$

Plan: signs → equation → d → 2 cars, 1 triangle

A toward → y shrinks → $\frac{dy}{dt} = -30$

B away → x grows → $\frac{dx}{dt} = +40$

(a)

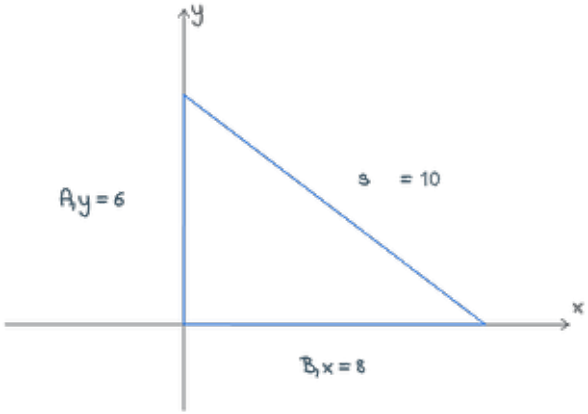
$$s^2 = x^2 + y^2$$

$$2s\,ds = 2x\,dx + 2y\,dy$$

$$\div 2\,dt \rightarrow s\frac{ds}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

Your turn: s → substitute

Solution The triangle at this moment, then the sign



Plan: s now → plug in → read the sign

$$s = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$$

$$10\frac{ds}{dt} = 8(40) + 6(-30) = 320 - 180 = 140$$

$$\frac{ds}{dt} = \frac{140}{10} = 14$$

(b)

$\frac{ds}{dt} > 0$ → distance increasing at 14 mph

Wrong sign +30 → $\frac{320 + 180}{10} = 50$ ✗

Workbook → each sign first

PROBLEM 3 OF 3

An angle of elevation

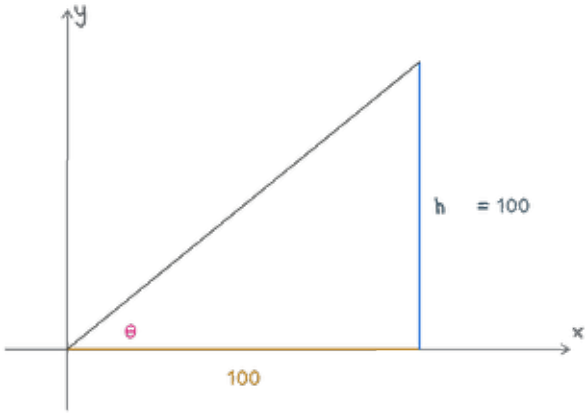
A balloon rises straight up at 5 m/s from a point 100 m away from an observer. How fast is the angle of elevation θ increasing when the balloon is 100 m high?

(A) $\frac{1}{20}$ rad per s

(B) $\frac{1}{40}$ rad per s

(C) $\frac{1}{10}$ rad per s

(D) $\frac{1}{100}$ rad per s



Solution Tangent, then differentials

Plan: opposite, adjacent → $\tan\theta = \frac{h}{100}$

$$\sec^2\theta\, d\theta = \frac{1}{100}\, dh$$

$$\div dt \rightarrow \sec^2\theta \frac{d\theta}{dt} = \frac{1}{100} \cdot \frac{dh}{dt}$$

$h = 100 \rightarrow \tan\theta = 1 \rightarrow \sec^2\theta = 1 + \tan^2\theta = 2$

$$2\frac{d\theta}{dt} = \frac{5}{100} \rightarrow \frac{d\theta}{dt} = \frac{1}{40} \text{ rad per s}$$

So (B).

Workbook → keep the $\sec^2$

WRAP-UP

Same speed? No

No → out 4, down 8

Fixed → number: the ladder doesn't stretch

$\frac{x}{y}$ grows → speeds up

Closer → negative

Workbook · video · comments · next 4.5: cones, shadows

Next: the workbook for this topic (link in the video description). Stuck? Ask in the comments.