3.4 / Differentiating Inverse Trigonometric Functions

Watch the lesson ➤

Derivatives of inverse sine, cosine, and tangent

Learning goal
Differentiate the inverse sine, cosine and tangent, alone and with an inside, and know why the formulas have no trig in them.

Key ideas

01 $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$ means $\sin\frac{\pi}{6} = \frac{1}{2}$, with angles in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ **only**. $\sin^{-1} x$ is not $\frac{1}{\sin x}$.

02 Flipping a graph over $y = x$ turns a slope m into $\frac{1}{m}$: at 0 the slope 1 stays 1; at the top a **flat** tangent flips to a wall.

03 $y = \sin^{-1} x$: $\cos y \frac{dy}{dx} = 1$, and from the triangle $\cos y = \sqrt{1 - x^2}$ (the **positive** root on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$). So $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$, for $-1 < x < 1$.

04 $y = \tan^{-1} x$: $\sec^2 y \frac{dy}{dx} = 1$, $\frac{dy}{dx} = \cos^2 y = \frac{1}{1 + x^2}$: always **rising**, slope 1 at 0.

05 $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, so $\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}}$: the same as $\sin^{-1} x$, with a **minus**.

Class notes

Remember

inverse sine
$\sin^{-1} x$: the angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is x (also written $\arcsin x$).

inverse tangent
$\tan^{-1} x$: the angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is x.

3.4 / Write it clearly

What earns the point on the AP Exam, and the mistakes that lose it.

Common mistakes

The mistake code is used in the answers and in My Mistakes

01 The inside's derivative goes on top

Code C1

✗ $\frac{d}{dx}\tan^{-1}(2x) = \frac{1}{1+4x^2}$.

✓ Top: the inside's derivative 2. $\frac{2}{1+4x^2}$.

3.4 / Guided examples

Watch, pause, and write the reasoning. Full solutions are in the video.

Problem types

Read the problem, name the type, then use its method

Type	Method	Problems
An inverse trig function of an inside u	Name the inside u and u' first; the inside's derivative goes on top: $\frac{u'}{1+u^2}$, $\frac{u'}{\sqrt{1-u^2}}$, $-\frac{u'}{\sqrt{1-u^2}}$.	E01 · Practice 1, 3, 8
Inverse trig inside another rule (a power, a sum, a product)	The last operation picks the first rule; the inverse trig formula is used inside it.	E02 · Practice 2, 4, 7
An inverse trig function as the inside of f, from given values	Inside value first ($\tan^{-1} 1 = \frac{\pi}{4}$), then f'(inside value) times the inside's derivative.	E03 · Practice 5
An inverse trig function in context, with units	The rate of the angle per foot: the sign says increasing or decreasing; units radians per foot.	Practice 6

Example 01

Video · E01

Let $y = \tan^{-1}(2x)$.

Write an equation of the tangent line to the graph at $x = \frac{1}{2}$.

> The point first, then the slope.

Example 02

Video · E02

If $f(x) = x \sin^{-1} x$, then $f'\left(\frac{1}{2}\right) =$

A. $\frac{\pi}{6}$

B. $\frac{\sqrt{3}}{3}$

C. $\frac{\pi}{6} + \frac{\sqrt{5}}{5}$

D. $\frac{\pi}{6} + \frac{\sqrt{3}}{3}$

A product: first′ times second, plus first times second′.

Example 03

Video · E03

The table gives values of a differentiable function f and its derivative.
(a) Let $h(x) = \tan^{-1}(f(x))$. Find $h'(3)$.
(b) Let $k(x) = \sin^{-1}(f(x))$. Find $k'(2)$.

x	1	2	3
$f(x)$	2	0	1
$f'(x)$	−1	3	4

The outside's rule, with the inside value from the table first.

VIDEO COMPLETE Check your example answers at the end of this booklet.

3.4 / Example answers

Differentiating Inverse Trigonometric Functions

E01 $y = \frac{\pi}{4} + \left(x - \frac{1}{2}\right)$

E02 (D)

E03 (a) 2; (b) 3