**Lecture Notes · Unit 3 · Lecture 5** AP CED 3.3

# Inverse Functions

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## INTRO: A FUNCTION THAT RUNS BACKWARD

Derivatives of Inverse Functions AP Calculus 3.3

$f(x) = x^3 + x + 2$ → backward: $g$ output → input

Swap $x, y$: $x = y^3 + y + 2$ → $y = ?$

No formula for $g$. Slope of $g$? Hold.

Formula, table, graph → slope of $g$

## IDEA 1: SWAP THE POINT

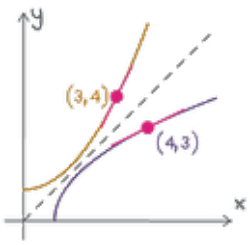
### Key ideas

1. An inverse swaps the coordinates

swap: $(a, b)$ on $f$ ⇔ $(b, a)$ on $g$

e.g. $f(x) = \frac{x^2}{3} + 1$ $(x \geq 0)$ → $f(3) = \frac{9}{3} + 1 = 4$.

So $g(4) = 3$.




2. Reflection flips the slope

on $f$ $\frac{\Delta y}{\Delta x}$ → on $g$ $\frac{\Delta x}{\Delta y}$

slope → $\frac{1}{\text{slope}}$ ?

Rise, run: traded places.

$f'(x) = \frac{2x}{3}$ → $f'(3) = 2$

So $g'(4) = \frac{1}{f'(3)} = \frac{1}{2}$, $2 \cdot \frac{1}{2} = 1$

## IDEAS 3–4: THE CHAIN RULE, THE RECIPE

### Key ideas

3. The same from the chain rule

$f(g(x)) = x$

Both sides → $\frac{d}{dx}$, if $g'$ exists

$f'(g(x)) \cdot g'(x) = \frac{d}{dx}x = 1$

$g'(x) = \frac{1}{f'(g(x))}$, $f'(g(x)) \neq 0$

Derivative of the inverse

Name: $g = f^{-1}$, $f^{-1} \neq \frac{1}{f}$

4. Recipe: partner, then flip

$f(a) = b$ → $g(b) = a$ → $g'(b) = \frac{1}{f'(a)}$

$(f^{-1})'(b) = \frac{1}{f'(a)}$

partner $a$ → flip

Not $\frac{1}{f'(b)}$ → $b$ = y-value of $f$

Why the ln x rule holds

unit 2 rule: $(\ln x)' = \frac{1}{x}$; ln undoes $e^x$

$y = \ln x$ → $e^y = x$ → $e^y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$

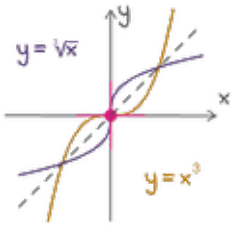
## IDEA 5: F′(A) ≠ 0

Key ideas

5. It needs $f'(a) \neq 0$

$f'(a) = 0 \rightarrow \frac{1}{0}$ undefined

$f'(0) = 0$




over $y = x \rightarrow \sqrt[3]{x} \rightarrow$ vertical tangent → no derivative.

$f(a) = b \rightarrow$ check $f'(a) \neq 0$ at a

## PROBLEM 1 OF 3

No formula for the inverse

Let $f(x) = x^3 + x + 2$, and let $g$ be the inverse function of $f$. What is $g'(4)$?

(A) $\frac{1}{49}$ (B) $\frac{1}{4}$ (C) 4 (D) 49

Inverse function

$g'(b) = \frac{1}{f'(a)}$, $f(a) = b$

First find a

$f(a) = b \rightarrow a = g(b)$

**Solution** Find a with f(a) = 4, then flip f'(a)

Plan: 4 = y of f → $f(a) = 4 \rightarrow \frac{1}{f'(a)}$

$a^3 + a + 2 = 4$

Cubic? → try a small a

$1 + 1 + 2 = 4 \rightarrow a = 1$

$g(4) = 1$, $(1, 4) \rightarrow (4, 1)$ on g

Step 2: your turn.

**Solution** Find a with f(a) = 4, then flip f'(a)

Inverse function

$g'(b) = \frac{1}{f'(a)}$, $f(a) = b$

First find a

$f(a) = b \rightarrow a = g(b)$

Plan: flip $f'(1)$

$f'(x) = 3x^2 + 1 \rightarrow f'(1) = 4$

$g'(4) = \frac{1}{f'(1)} = \frac{1}{4}$

(B) No formula for g → slope ✓

AP MC: Unit 3 → 5-10%

Workbook: find the partner first.

Careful the wrong point

X Common mistake

$\frac{1}{f'(4)}$

$= \frac{1}{49}$

✓ Do this

4 = y-value of f → a = 1

$g'(4) = \frac{1}{f'(1)} = \frac{1}{4}$

## PROBLEM 2 OF 3

An inverse from a table

The function $f$ is differentiable and has an inverse function $g$. The table gives values of $f$ and $f'$.

(a) Find $g(5)$.

(b) Find $g'(5)$.

(c) Write an equation of the tangent line to the graph of $g$ at $x = 5$.

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $f(x)$ | 3 | 5 | 6 | 1 |
| $f'(x)$ | 3 | $\frac{2}{3}$ | 4 | 5 |

**Solution** Find 5 in the f(x) row, then flip f'

Plan: 5 = y-value of f → f(x) row (not the x row)

(a)

$f(2) = 5 \rightarrow g(5) = 2$

(b)

$$g'(5) = \frac{1}{f'(g(5))} = \frac{1}{f'(2)}$$

$$= \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

(c): your turn.

**Solution** Find 5 in the f(x) row, then flip f'

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $f(x)$ | 3 | 5 | 6 | 1 |
| $f'(x)$ | 3 | $\frac{2}{3}$ | 4 | 5 |

Plan: (5, 2) from (a), slope $\frac{3}{2}$ from (b)

(c)

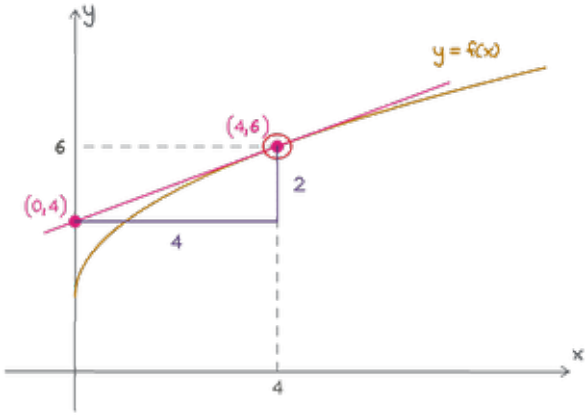
$$y = 2 + \frac{3}{2}(x - 5)$$

Workbook: the f(x) row.

## PROBLEM 3 OF 3

An inverse from a graph

The graph of $f$ is shown, with its tangent line at the point $(4, 6)$. The tangent line also passes through $(0, 4)$. Let $g$ be the inverse function of $f$. Find $g'(6)$.




**Solution** Read the point, then the line's slope, then flip

Plan: 6 = y-value of f → point → slope → flip

$f(4) = 6 \rightarrow g(6) = 4$

$$f'(4) = \text{slope} = \frac{6 - 4}{4 - 0} = \frac{1}{2}$$

$$g'(6) = \frac{1}{f'(4)} = \frac{1}{\frac{1}{2}} = 2$$

Not $\frac{1}{2}$ → slope of f

Workbook: tangent line

## WRAP-UP

No formula for g? Still a slope

Yes: $g'(4) = \frac{1}{f'(1)} = \frac{1}{4}$

y = x: swap the point, flip the slope

$f(a) = b$ first $\rightarrow g'(b) = \frac{1}{f'(a)}$

only if $f'(a) \neq 0$

workbook → comments → next: 3.4 inverse trig

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Next: the workbook for this topic (link in the video description). Stuck? Ask in the comments.