3.3 / Differentiating Inverse Functions

The derivative of an inverse function

Watch the lesson ➤

Learning goal

Find the derivative of an inverse function at a point without its formula: find a with f(a) = b, then flip f'(a).

Key ideas

01 $f(x) = x^3 + x + 2$ has an inverse g but no formula. Slope of g? An inverse swaps the coordinates: (a, b) on $f \Leftrightarrow (b, a)$ on g. E.g. $f(x) = \frac{x^2}{3} + 1$ $(x \geq 0)$: $f(3) = 4$, so $g(4) = 3$. The graph of g is the **reflection** of f over the line $y = x$.

02 The reflection flips the slope: rise and run trade places. $f'(x) = \frac{2x}{3}$, $f'(3) = 2$, so $g'(4) = \frac{1}{f'(3)} = \frac{1}{2}$.

03 Why: $f(g(x)) = x$, so $f'(g(x)) \cdot g'(x) = 1$ and $g'(x) = \frac{1}{f'(g(x))}$: the derivative of an **inverse** function.

04 Step 1: find a with $f(a) = b$. Step 2: flip $f'(a)$. **Not** $\frac{1}{f'(b)}$: b is a y-value of f.

05 Why $(\ln x)' = \frac{1}{x}$: $y = \ln x$ means $e^y = x$, so $e^y \frac{dy}{dx} = 1$ and $\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$.

06 It needs $f'(a) \neq 0$: $y = x^3$ has a horizontal tangent at 0; reflected over $y = x$, $y = \sqrt[3]{x}$ has a **vertical** tangent at 0, no derivative.

Remember

inverse function

$g = f^{-1}$: $g(b) = a$ exactly when $f(a) = b$; its graph is the reflection of f over $y = x$.

Class notes

3.3 / Write it clearly

What earns the point on the AP Exam, and the mistakes that lose it.

Common mistakes

The mistake code is used in the answers and in My Mistakes

01 Flip f' at a, not at b

Code V1

✗ $g = f^{-1}$, $f(x) = x^3 + x + 2$: $g'(4) = \dfrac{1}{f'(4)} = \dfrac{1}{49}$.

✓ 4 is a y-value of f. $f(1) = 4$, so use $a = 1$: $g'(4) = \dfrac{1}{f'(1)} = \dfrac{1}{4}$.

3.3 / Guided examples

Watch, pause, and write the reasoning. Full solutions are in the video.

Problem types

Read the problem, name the type, then use its method

Type	Method	Problems
$g'(b)$ for $g = f^{-1}$ from a formula for f	Find a with $f(a) = b$ first (try a small number), then flip: $g'(b) = \dfrac{1}{f'(a)}$.	E01, E03 · Practice 1, 2
$g'(b)$ from given values, a table, or a graph	Find b among the y-values of f; its x is $a = g(b)$. Then $\dfrac{1}{f'(a)}$, never $\dfrac{1}{f'(b)}$.	E02 · Practice 3, 4
When $f'(a) = 0$ (the reflected tangent is vertical)	A horizontal tangent of f at (a, b) reflects to a vertical tangent of g at (b, a); $g'(b)$ does not exist.	Practice 5
An inverse function in context, with units	The input of g is the output of f: the units of g' are flipped (minutes per gallon).	Practice 6

Example 01

Video · E01

Let $f(x) = x^3 + x + 2$, and let g be the inverse function of f. What is $g'(4)$?

A. $\dfrac{1}{49}$

B. $\dfrac{1}{4}$

C. 4

D. 49

> Find a with $f(a) = 4$ first. Then flip $f'(a)$.

Example 02

Video · E02

f is differentiable and has an inverse function g. Values of f and f' are in the table.
(a) Find $g(5)$.
(b) Find $g'(5)$.
(c) Write an equation of the tangent line to the graph of g at $x = 5$.

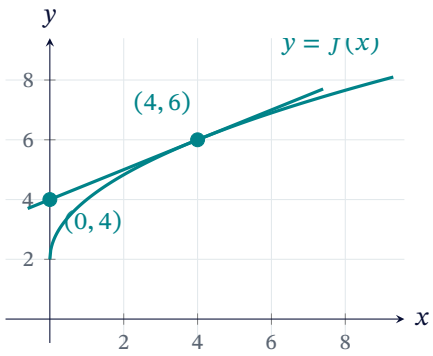
x	1	2	3	4
$f(x)$	3	5	6	9
$f'(x)$	3	$\frac{2}{3}$	4	5

> Look in the $f(x)$ column for 5.

Example 03

Video · E03

The graph of f is shown, with its tangent line at the point $(4, 6)$. The tangent line also passes through $(0, 4)$. Let g be the inverse function of f. Find $g'(6)$.



> Plan: 6 = a y-value → the point → slope → flip

VIDEO COMPLETE Check your example answers at the end of this booklet.

3.3 / Example answers

Differentiating Inverse Functions

E01 (B)

E02 (a) 2; (b) $\frac{3}{2}$; (c) $y = 2 + \frac{3}{2}(x - 5)$

E03 2