Lecture Notes · Unit 3 · Lecture 3 AP CED 3.2a

Implicit differentiation (1): finding dy/dx

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INTRO: CAN'T SOLVE FOR Y?

Implicit Differentiation AP Calculus 3.2

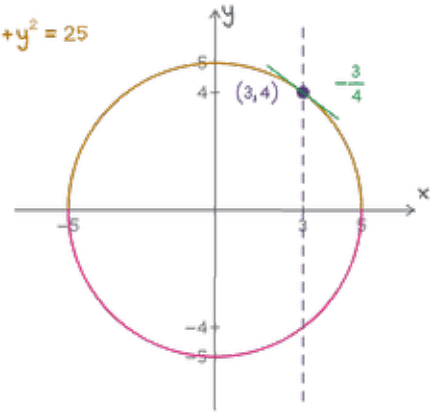
So far: y handed to us as a function of x

$x^2y+y^3 = 10 \to y = ?$

$\frac{dy}{dx}$ without solving for y

A CIRCLE: THE LONG WAY, THE FAST WAY

Slope of a circle



$x^2+y^2 = 25 \to$ a circle, radius 5

vertical line hits twice → not a function

(3, 4) → slope?

The long way: solve for y

$y^2 = 25-x^2 \to y = \pm\sqrt{25-x^2}$

Top: $y = \sqrt{25-x^2}$

Bottom: $y = -\sqrt{25-x^2}$

(3, 4) on top → outside $\sqrt{u}$, inside $25-x^2$

$\sqrt{u} = u^{\frac{1}{2}} \to y' = \frac{1}{2}(25-x^2)^{-\frac{1}{2}} \cdot (-2x)$

$= -\frac{2x}{2\sqrt{25-x^2}}$

$y'(3) = \frac{-3}{\sqrt{16}} = -\frac{3}{4}$

It works, but we had to solve for y and pick a half

The fast way: differentiate both sides

$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(25)$

$\frac{d}{dx}(x^2) = 2x$

$y = y(x)$: a hidden function, a box □

y^2: outside u^2, inside y

$u = y \quad \frac{d}{dx}(u^2) = 2u\frac{du}{dx} = 2y\frac{dy}{dx}$

$\frac{dy}{dx}$ = the inside's derivative → chain rule

BY HAND: TAG WITH D, DIVIDE BY DX

Implicit differentiation by hand

By hand: tag every letter with its d

$x^2 \to 2x\,dx, \quad x^3 \to 3x^2\,dx$

$y^2 \to 2y\,dy$

divide by dx: $2x\frac{dx}{dx} = 2x \cdot 1 = 2x$

$2y\frac{dy}{dx} \to \frac{dy}{dx}$ stays

only y in it: differentiate w.r.t. y, then $\cdot\frac{dy}{dx}$

Three quick examples

$(fg)' = f'g+fg', \qquad \left(\frac{f}{g}\right)' = \frac{f'g-fg'}{g^2}$

(1) $y = f(x)$: $1\,dy = f'(x)\,dx$

$\rightarrow \frac{dy}{dx} = f'(x)$

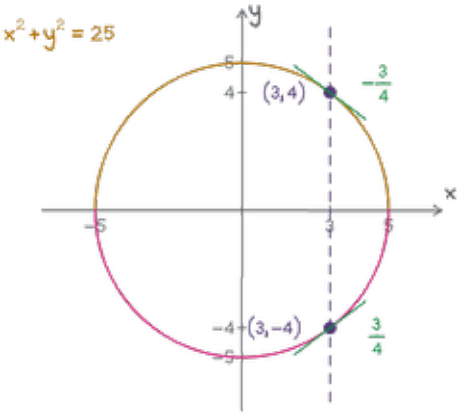
$y = f(t) \rightarrow 1\,dy = f'(t)\,dt \rightarrow$ need $\frac{dy}{dt}$? Divide by dt

(2) $x^2y \rightarrow 2x\,dx \cdot y + x^2 \cdot 1\,dy$

$\div dx \rightarrow 2xy + x^2\frac{dy}{dx}$

(3) $\frac{x}{y} \rightarrow \frac{1\,dx \cdot y - x \cdot 1\,dy}{y^2}$

$\div dx \rightarrow \frac{y - x\frac{dy}{dx}}{y^2}$



Back to the circle

$x^2 + y^2 = 25 \rightarrow 2x + 2y\frac{dy}{dx} = 0$

$2y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = -\frac{x}{y}$

The bottom half

$\frac{dy}{dx} = -\frac{x}{y}$

At $(3,-4)$: $-\frac{3}{-4} = \frac{3}{4}$

No half to choose → the formula knows

Implicit: y stays inside. Explicit: y = formula in x

EVERY Y TERM; THE FOUR MOVES

Every y term, four moves

Every y term gets a tag

$\frac{d}{dx}(y^3) = 3y^2\frac{dy}{dx}$

$\sin y \rightarrow \cos y \cdot \frac{dy}{dx}$

$e^y \rightarrow e^y\frac{dy}{dx}$

$(fg)' = f'g + fg' \Rightarrow xy \rightarrow 1 \cdot y + x\frac{dy}{dx}$

$x^3 \rightarrow 3x^2$, no $\frac{dy}{dx}$ (w.r.t. x)

Four moves

1. Differentiate both sides.

d's → divide by dx: no bare dy

Keep the = and its right side

2. $\frac{dy}{dx}$ terms → one side

3. Factor. 4. Divide.

PROBLEM 1 OF 2

dy/dx at a point, y not solved

A curve is given by $x^2y + y^3 = 10$.

Find $\frac{dy}{dx}$ at the point $(3, 1)$.

Product rule
$(fg)' = f'g + fg'$
Each y term
$\frac{d}{dx}y^3 = 3y^2\frac{dy}{dx}$

Solution Differentiate both sides

Plan: $(3, 1)$ on the curve? → both sides

$3^2(1) + 1^3 = 9 + 1 = 10$

$\frac{d}{dx}(x^2y)\ (f = x^2,\ g = y)$

$$= 2x \cdot \frac{dx}{dx} \cdot y + x^2 \cdot \frac{dy}{dx}$$

$$2xy + x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

Next three moves: yours.

Solution Move, factor, divide, then the point

Product rule
$(fg)' = f'g + fg'$
Each y term
$\frac{d}{dx}y^3 = 3y^2\frac{dy}{dx}$

Plan: move → factor → divide → (3, 1)

$$x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx} = -2xy$$

$$\left(x^2 + 3y^2\right)\frac{dy}{dx} = -2xy$$

$$\frac{dy}{dx} = \frac{-2xy}{x^2 + 3y^2}$$

$$\frac{-2(3)(1)}{9+3} = \frac{-6}{12} = -\frac{1}{2}$$

Workbook: one more curve.

Careful x^2y is a product

X Common mistake

$x^2y' + 3y^2y' = 0$

$0 = 12y'$

$y' = 0$?!

✓ Do this

$$2xy + x^2\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{2}$$

2023 AP FRQ 6 → (a) 2 of 9 points → mean 2.77 of 9

PROBLEM 2 OF 2

A slope with e to the y

If $e^y + xy = e$, what is the value of $\frac{dy}{dx}$ at the point $(0, 1)$?

(A) $-\frac{1}{e}$ (B) $-e$ (C) $\frac{1}{e}$ (D) 0

Each y term
$\frac{d}{dx}e^y = e^y\frac{dy}{dx}$
Product rule
$(fg)' = f'g + fg'$
$\frac{d}{dx}(xy) = 1 \cdot y + x\frac{dy}{dx}$

Solution Both sides, then plug in the point

Plan: both sides → plug in (0, 1) → no general $\frac{dy}{dx}$

$$e^y\frac{dy}{dx} + y + x\frac{dy}{dx} = 0$$

$$x = 0,\ y = 1 \rightarrow e^1\frac{dy}{dx} + 1 + 0 \cdot \frac{dy}{dx} = 0$$

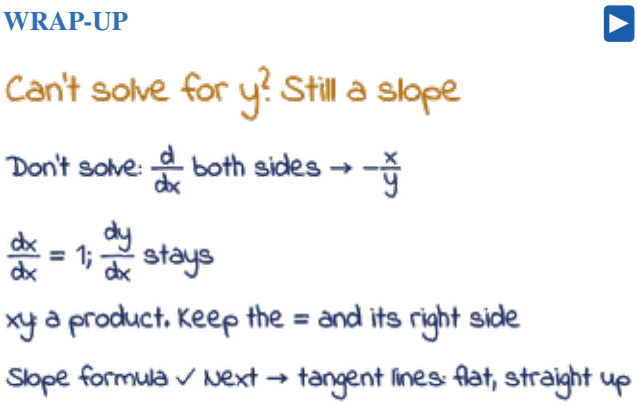
$$e\frac{dy}{dx} = -1$$

$$\frac{dy}{dx} = -\frac{1}{e}$$

So (A).

Workbook: one more with e^y.

WRAP-UP



Next: the workbook for this topic (link in the video description). Stuck? Ask in the comments.