Lecture Notes · Unit 3 · Lecture 1 AP CED 3.1a

The chain rule

Tap a blue play button next to a heading to open that part of the video. ▶ Watch the video

INTRO: IS IT JUST COS 2X?

The Chain Rule AP Calculus 3.1

$(\sin x)' = \cos x$

just $\cos 2x$? → test it

See why the inside′ multiplies

Layers → peel them, keep every inside′

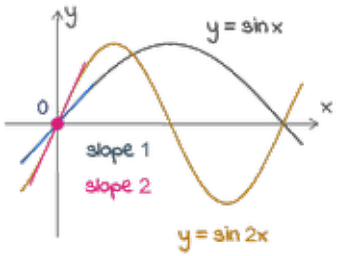
Let's get that 5

SAME WAVE, TWICE AS FAST

Key ideas

Same wave, twice as fast

$\sin x$, $\sin 2x$



At 0 → slopes 1 and 2

Inside $2x$ → 2 x as fast

Guess → $2\cos 2x$ → why?

INSIDE AND OUTSIDE

Key ideas

Inside and outside

$(3x+1)^2$ → 2 jobs

First the inside → $u = 3x+1$

Then the outside → $y = u^2$

$y = f(g(x))$ → g inside first, f outside last

Inside → what you'd compute first

$\sin^2 x = (\sin x)^2$ → $\sin x$ inside · $\sin(x^2)$ → x^2 inside

Inside just x → times 1

$(3x+1)^2 = 9x^2+6x+1$

$\frac{d}{dx}(\cdots) = 18x+6$

$18x+6 = 2(3x+1)\cdot 3$

$2(3x+1)$ → outside′, inside alone. $\cdot 3$ → inside′.

$(x^2+1)^2 = x^4+2x^2+1$

$\frac{d}{dx}(\cdots) = 4x^3+4x$

$4x^3+4x = 2(x^2+1)\cdot 2x$

Extra $2x$ → not a number → inside′

Guess: $\frac{d}{dx}(x^2+1)^3 = 3(x^2+1)^2 \cdot 2x$

$(x^2+1)^3 = x^6+3x^4+3x^2+1$

$\frac{d}{dx}(\cdots) = 6x^5+12x^3+6x$

Guess → $6x^5+12x^3+6x$ ✓

WHY MULTIPLY? FROM THE DEFINITION

Key ideas

Why multiply? Straight from the definition

$y = f(g(x))$

$y' = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}$

$\triangle = g(x+h)$, $\square = g(x)$

$= \lim_{h \to 0} \frac{f(\triangle) - f(\square)}{h}$

Multiply and divide by $\triangle - \square$, moves $\to \neq 0$

$= \lim_{h \to 0} \frac{f(\triangle) - f(\square)}{\triangle - \square} \cdot \frac{g(x+h) - g(x)}{h}$

$h \to 0 \to \triangle \to \square$, g differentiable $\Rightarrow$ continuous

1st $\to$ like $\frac{f(t) - f(a)}{t - a}$, $t \to a$; $t = \triangle$, $a = \square \to$
$f'(\square) = f'(g(x))$

2nd $\to$ definition of $g'(x)$

$= f'(g(x)) \cdot g'(x)$

NAME THE INSIDE: THE CHAIN RULE

Key ideas

Name the inside u

$u = g(x)$, $y = f(u)$

$f'(g(x)) = \frac{dy}{du}$, $g'(x) = \frac{du}{dx}$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \to$ rates multiply

$u = 3x + 1$, $y = u^2 \to \frac{dy}{du} = 2u$, $\frac{du}{dx} = 3$
$6(3x + 1) = 18x + 6$

The chain rule

$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

outside'(inside kept) x inside' $\to$ outside first

$(\sin 2x)' = \cos 2x \cdot 2 = 2\cos 2x \to$ same at 0

BOX SHAPES AND LAYERS

Key ideas

Shapes you'll see all the time

$\frac{d}{dx}(\square)^n = n(\square)^{n-1} \cdot \square'$

$\frac{d}{dx} e^{\square} = e^{\square} \cdot \square'$

$\frac{d}{dx} \ln \square = \frac{\square'}{\square} (\square > 0)$

$\frac{d}{dx} \sin \square = \cos \square \cdot \square'$

$\frac{d}{dx} \cos \square = -\sin \square \cdot \square'$, $\tan \square \to \sec^2 \square \cdot \square'$

$(\sqrt{\square})' = \frac{\square'}{2\sqrt{\square}} (\square > 0)$

More layers: peel one at a time

outside in $\to$ $y = e^{\sin(4x)} \to e$, sin, $4x$

$e \to e^{\sin(4x)}$

$\sin \to \cos(4x)$

$4x \to 4$

3 links $\to$ $y' = e^{\sin(4x)} \cdot \cos(4x) \cdot 4$

$= 4\cos(4x) e^{\sin(4x)} \to$ a chain

ANY BASE A

Key ideas

Any base $a > 0$

base e: $e^{\ln u} = u \to a^x = e^{\ln(a^x)}$

Exponent to the front $\to$ $a^x = e^{x \ln a}$

$e^{\text{inside}} \to \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \cdot \ln a \to$ inside: $x \cdot \ln a$

So $\frac{d}{dx} a^x = a^x \ln a$. e.g. $\frac{d}{dx} 2^x = 2^x \ln 2$.

$a = e \to \ln e = 1 \to e^x$ stays e^x ✓

A log of any base

$\log_a x = \frac{\ln x}{\ln a} (a \neq 1) \to \frac{d}{dx} \log_a x = \frac{1}{x \ln a}$

$\log_2 x \to \frac{1}{x \ln 2}$

PROBLEM 1 OF 3

A tangent line to a composite

Let $f(x)=\sqrt{x^2+5}$.

(a) Find $f'(x)$.

(b) Write an equation of the tangent line to the graph of f at $x=2$.

Chain rule

$$\frac{d}{dx}f(g(x))=f'(g(x))g'(x)$$

Solution (a) The chain rule

Plan: outside $\sqrt{\ }$, inside x^2+5 → chain rule

(a)

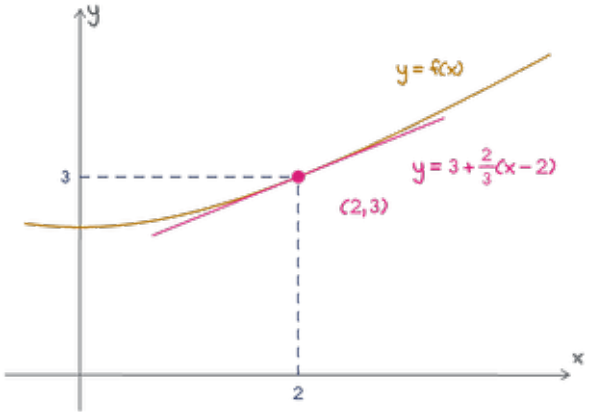
$$f(x)=(x^2+5)^{\frac{1}{2}}$$

$$f'(x)=\frac{1}{2}(x^2+5)^{-\frac{1}{2}}\cdot 2x$$

$$=\frac{2x}{2\sqrt{x^2+5}}=\frac{x}{\sqrt{x^2+5}}$$

Your turn: part (b), $x=2$.

Solution (b) The point and the slope



(b)

Plan: point $f(2)$, slope $f'(2)$

$$f(2)=\sqrt{2^2+5}=\sqrt{9}=3$$

$$f'(2)=\frac{2}{\sqrt{2^2+5}}=\frac{2}{3}$$

Tangent line: $y=3+\frac{2}{3}(x-2)$

Workbook: one like this.

Careful the inside's derivative (2025 AP, Q5)

X Common mistake

$$\frac{1}{2}(x^2+5)^{-\frac{1}{2}}$$

$$f'(2)=\frac{1}{6}$$

✓ Do this

x inside' $2x$

$$\frac{x}{\sqrt{x^2+5}}$$

$$f'(2)=\frac{2}{3}$$

PROBLEM 2 OF 3

A horizontal tangent

Let $h(x)=x^2e^{-3x}$. At which x-values does the graph of h have a horizontal tangent line?

(A) 0 only (B) $\frac{2}{3}$ only

(C) 0 and $\frac{2}{3}$ (D) 0 and -2

Chain rule

$$\frac{d}{dx}f(g(x))=f'(g(x))g'(x)$$

Product rule

$$\frac{d}{dx}(uv)=u'v+uv'$$

Solution Product rule, a chain inside

Plan: $h'(x)=0$ → product $x^2\cdot e^{-3x}$, chain e^{-3x}

$\frac{d}{dx}e^{-3x} = e^{-3x} \cdot (-3)$ inside′

$h'(x) = 2xe^{-3x} + x^2 \cdot (-3e^{-3x})$

$= xe^{-3x}(2 - 3x)$, $e^{-3x} > 0$

$x = 0$ or $x = \frac{2}{3}$

So (C).

D: 0, −2 → −3 missing → trap

Your turn: workbook, one like it.

PROBLEM 3 OF 3

Three layers

If $f(x) = \sin^2(3x)$, then $f'\left(\frac{\pi}{12}\right) =$

(A) 1 (B) $\frac{3}{2}$ (C) 3 (D) $3\sqrt{2}$

Chain rule

$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

Shapes

$\frac{d}{dx}(\square)^2 = 2\square \cdot \square'$

$\frac{d}{dx}\sin\square = \cos\square \cdot \square'$

Solution Peel the layers: square, sine, 3x

Plan: $[\sin(3x)]^2$ → three links: $(\,)^2$, sin, 3x

$f'(x) = 2\sin 3x \cdot \cos 3x \cdot 3$

$= 6\sin 3x\cos 3x$

$x = \frac{\pi}{12} \rightarrow 3x = \frac{\pi}{4}$.

$f'\left(\frac{\pi}{12}\right) = 6 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = 6 \cdot \frac{2}{4}$

$= 3 \rightarrow$ (C).

Workbook: one more.

WRAP-UP

Just $\cos 2x$? No: $2\cos 2x$

No → $(\sin 2x)' = 2\cos 2x$, not $\cos 2x$

$f'(g(x)) \cdot g'(x)$ → the inside multiplies

Why △ → □ → two fractions → two derivatives

More layers → outside in, link by link

Next → 3.1, tables and graphs

Next: the workbook for this topic (link in the video description). Stuck? Ask in the comments.