Lecture Notes · Unit 2 · Lecture 5 AP CED 2.5–2.6

Power rule, constants, sums

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INTRO: THE SLOPE OF X TO THE TENTH

The Power Rule AP Calculus 2.5-2.6

2.4: f′ breaks → today: f′ exists

Slope of x^{10} at 1?

$(1+h)^{10}$ → 11 terms

$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$

No expanding → factor the top

Today: one line, and why

SQUARES AND CUBES, BY THE DEFINITION

x^2 and x^3: factor, cancel, let $x \to a$

$\lim_{x \to a} \frac{x^2-a^2}{x-a}$

$= \lim_{x \to a} \frac{(x-a)(x+a)}{x-a}$ $(x \neq a)$

$= a+a = 2a$

$\lim_{x \to a} \frac{x^3-a^3}{x-a}$

$= \lim_{x \to a} \frac{(x-a)(x^2+xa+a^2)}{x-a}$

$= a^2+a^2+a^2 = 3a^2$

THE PATTERN: GUESS, THEN CHECK

$x \to a$, factor (cont.): the pattern

$2a,\ 3a^2 \to x^4$: ?

Guess → $4a^3$

$x^4-a^4 = (x^2-a^2)(x^2+a^2)$

$= (x-a)(x+a)(x^2+a^2)$

$\lim_{x \to a} (x+a)(x^2+a^2)$

$= 2a \cdot 2a^2 = 4a^3$ ✓

$2a,\ 3a^2,\ 4a^3 \to$ guess: na^{n-1}

$x^n - a^n = (x-a)(x^{n-1}+x^{n-2}a+\cdots+a^{n-1})$

Bracket: x^{n-1} down to x^0 → n terms

$\to a^{n-1}+\cdots+a^{n-1} = (n)a^{n-1} = na^{n-1}$

Guess ✓ → the power rule

THE POWER RULE

$\frac{d}{dx}x^n = nx^{n-1}$, n a positive integer

Exponent down, then −1

$x^{10} \to 10x^9$ → at 1: 10

Any real r: $\frac{d}{dx}x^r = rx^{r-1}$, where defined

$\sqrt{x} = x^{\frac{1}{2}} \to \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$, $x > 0$

= 2.2's result by the definition ✓

$\frac{1}{x} = x^{-1} \to -x^{-2} = -\frac{1}{x^2}$, $x \neq 0$

$\frac{1}{x^2} = x^{-2} \to -2x^{-2-1}$, $x \neq 0$

Careful: not $-1 \to -2-1 = -3 \to -\frac{2}{x^3}$

REWRITE FIRST

The power rule (cont.): rewrite first

The rule wants x^r → rewrite first

$\sqrt[3]{x} = x^{\frac{1}{3}} \to \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$

$x = 0$: bottom 0 → no $f'(0)$ (2.4)

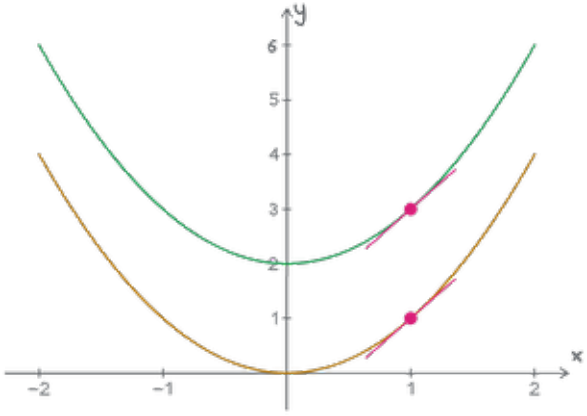
$\frac{x^3+1}{x} = \frac{x^3}{x} + \frac{1}{x} = x^2 + x^{-1},\ x \neq 0$

Every piece x^r → the rule

CONSTANTS AND CONSTANT MULTIPLES

Constants (2.6)

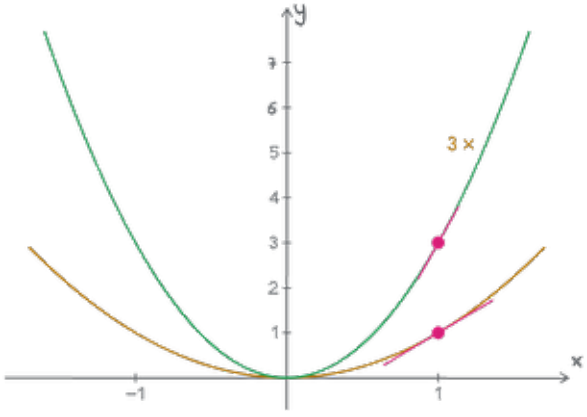
Shift x^2 up 2



$(x^2+2)' = 2x+0$

$\frac{d}{dx}c = 0$

π^3, e^2: numbers alone → 0



$(3x^2)' = 3 \cdot 2x = 6x$

$\frac{d}{dx}[c\,f(x)] = c\,f'(x)$

Example: $\frac{x^2}{5}$

$\frac{x^2}{5} = \frac{1}{5}x^2 \to \frac{1}{5} \cdot 2x = \frac{2}{5}x$

Alone → 0. In front → stays.

SUMS AND DIFFERENCES; A GIVEN SLOPE

Sums and differences (2.6)

$f+g \to \frac{f(x)-f(a)}{x-a} + \frac{g(x)-g(a)}{x-a}$

Limit law → $(f \pm g)' = f' \pm g'$

$(x^4 - 6x^2 + 2)' = 4x^3 - 6(2x) + 0 = 4x^3 - 12x$

$\left(\sum_{k=1}^{n} f_k\right)' = \sum_{k=1}^{n} f'_k$

only ±; ×, ÷ → 2.8, 2.9

Given slope k → solve $f'(x) = k$

Horizontal → $f'(x) = 0$

Parallel → same slope → $f'(x) = g'(x)$

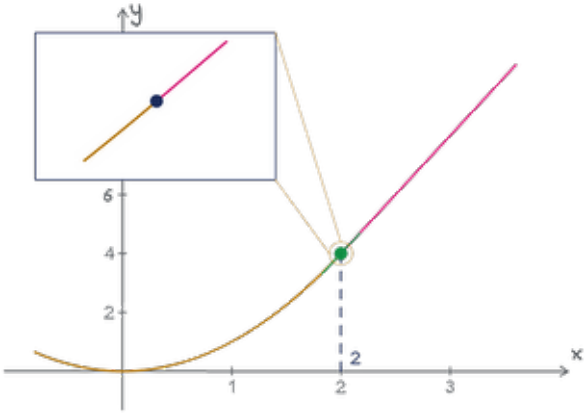
e.g. $f(x) = x^3$, $g(x) = 3x^2 + 1$ → $3x^2 = 6x$

$3x^2 - 6x = 0 \Rightarrow 3x(x-2) = 0$

Parallel at $x = 0$, $x = 2$

WHERE TWO PIECES MEET

Pieces meet → heights, then slopes



x^2 $(x \leq 2)$, $4x-4$ $(x > 2)$

(i) $2^2 = 4$, $4(2) - 4 = 4$ → f continuous at 2

2.4 V → same height, corner

(ii) $2x \to 2(2) = 4$, $(4x-4)' = 4$ → f differentiable at 2

2.4 slopes: $3x - 1 \to 3$, $x^2 - 2x + 5 \to 2$

PROBLEM 1 OF 3

Rewrite first

Let $f(x)=\dfrac{x^3-4\sqrt{x}}{x}$. What is the value of $f'(4)$?

(A) 47 (B) 7 (C) 12 (D) $\dfrac{33}{4}$

Power rule
$\frac{d}{dx}x^r = rx^{r-1}$
Rewrite
$\sqrt{x} = x^{\frac{1}{2}}$, $x = x^1$
Constant multiple
$(cf)' = cf'$

Solution Rewrite as powers of x, then the power rule

Plan: powers of x → the rule, term by term

$$f(x) = \frac{x^3}{x} - \frac{4x^{\frac{1}{2}}}{x}$$

$$= x^2 - 4x^{-\frac{1}{2}},\ x > 0$$

$$f'(x) = 2x^{2-1} - 4\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1}$$

$$= 2x + 2x^{-\frac{3}{2}}$$

$$f'(4) = 2(4) + 2 \cdot 4^{-\frac{3}{2}}$$

$$4^{-\frac{3}{2}} = \frac{1}{(\sqrt{4})^3} = \frac{1}{2^3} = \frac{1}{8}$$

$$f'(4) = 8 + \frac{1}{4} = \frac{33}{4} \rightarrow \text{So (D)}$$

Workbook: rewrite, then the rule

COMMON MISTAKE: TOP PRIME OVER BOTTOM PRIME

Why is (A) 47 there?

Why is (A) 47 there?

$$\frac{(\text{top})'}{(\text{bottom})'} = \frac{3x^2 - 2x^{-\frac{1}{2}}}{1}$$

At 4: $3(16) - 2 \cdot 4^{-\frac{1}{2}} = 48 - 1 = 47$

Dividing derivatives: not a rule → (A) X

Rewrite → $f'(4) = \frac{33}{4}$

PROBLEM 2 OF 3

Tangent lines

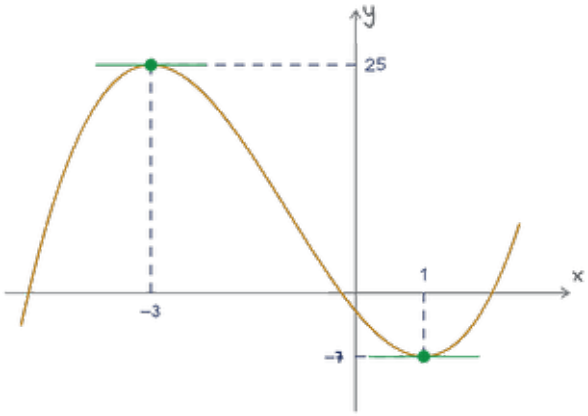
Let $f(x)=x^3+3x^2-9x-2$.

(a) Find $f'(x)$.

(b) Find the points with a horizontal tangent.

(c) Write the tangent line at $x=0$.

(d) Find the other x with a parallel tangent.



Solution (a)(b) Term by term; horizontal = slope 0

Plan: term by term → point from f, slope from f'

(a)

$$f'(x) = 3x^2 + 3(2x) - 9(1) - 0$$

$= 3x^2 + 6x - 9$

(b)

$f'(x) = 0 \Rightarrow 3x^2 + 6x - 9 = 0$

$3(x+3)(x-1) = 0 \Rightarrow x = -3$ or $x = 1$

$f(-3) = (-3)^3 + 3(-3)^2 - 9(-3) - 2$

$= -27 + 27 + 27 - 2 = 25$

$f(1) = 1 + 3 - 9 - 2 = -7$

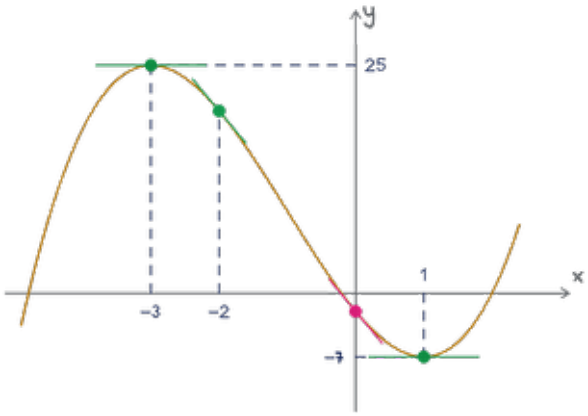
$(-3, 25),\ (1, -7)$

(c)

Tangent at $x = 0$ · (d) parallel

Solution

(c)(d) Point from f, slope from f'; parallel = the same slope



(c)

$f(0) = -2$

$f'(0) = 3(0)^2 + 6(0) - 9 = -9$

$y = f(0) + f'(0)(x - 0)$

$= -2 - 9(x - 0) = -2 - 9x$

(d)

Parallel → slope -9

$f'(x) = -9 \Rightarrow 3x^2 + 6x - 9 = -9$

$3x^2 + 6x = 0 \Rightarrow 3x(x+2) = 0$

$x = 0$ (given), $x = -2$

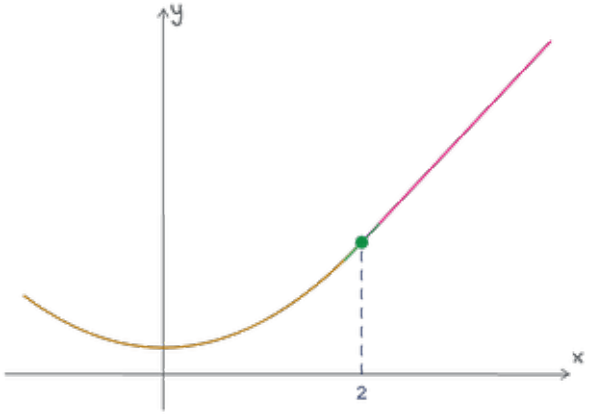
Workbook: slope 0 or same slope

PROBLEM 3 OF 3

Two constants

Let $f(x) = \begin{cases} ax^2 + 1, & x \le 2 \\ bx - 3, & x > 2 \end{cases}$, where a and b are constants.

Find a and b so that f is differentiable at $x = 2$. Justify your answer.



Solution

Heights match (continuous), then slopes match

Plan: 2 unknowns → heights, slopes

Differentiable at 2 ⇒ continuous at 2

Both slopes use $f(2)$ → heights first

(i) $a(2)^2 + 1 = b(2) - 3 \rightarrow 4a + 1 = 2b - 3$

$(ax^2 + 1)' = 2ax,\ (bx - 3)' = b$

(ii) $2a(2) = b \Rightarrow b = 4a$

$4a + 1 = 2(4a) - 3 = 8a - 3$

$4 = 4a \Rightarrow a = 1$

$a = 1, b = 4$

Check: $1 \cdot 4 + 1 = 5$, $4 \cdot 2 - 3 = 5$; slopes 4, 4

Workbook: two constants

WRAP-UP

In one line

The n in front counts the terms

Real r: only where defined

Alone → 0; $(c\,f)' = c\,f'$

Term by term; parallel: $f' = g'$

Joints: heights, then slopes

Next → 2.7 → $\sin x$, $\cos x$, e^x, $\ln x$

Next: the workbook for this topic (link in the video description). Stuck? Ask in the comments.