Lecture Notes · Unit 2 · Lecture 4 AP CED 2.4

Differentiability and continuity

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IDEA 1: A DERIVATIVE FORCES CONTINUITY

Where a Derivative Fails AP Calculus 2.4

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Real limit → what about f?

Forward ✓, backward → a trap

f'(x) exists → real number

Bottom h → 0

Top → 5? → $\left|\frac{\text{near } 5}{\text{near } 0}\right| \to \infty$

Limit exists → so top → 0

$$\lim_{h \to 0} [f(x+h) - f(x)] = 0$$

$$\lim_{h \to 0} f(x+h) = f(x)$$

$x + h = t \to h \to 0 \Leftrightarrow t \to x$

$$\lim_{t \to x} f(t) = f(x) \to \text{continuous at } x$$

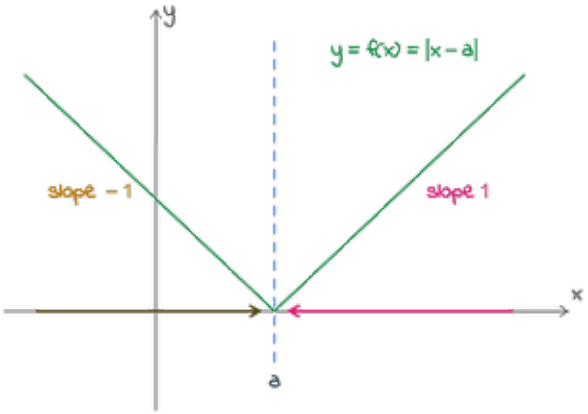
Every x → a continuous function

Differentiable ⇒ continuous ✓

"f differentiable at c, so continuous at c."

IDEA 2: CONTINUOUS, BUT NO SLOPE

Draw |x − a|



x = a instant → limit, rate → slope

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$\because f(a) = |a - a| = 0$

$$= \lim_{x \to a} \frac{|x-a|}{x-a}$$

Right → $x > a \to |x - a| = x - a$

$$\frac{x-a}{x-a} = 1$$

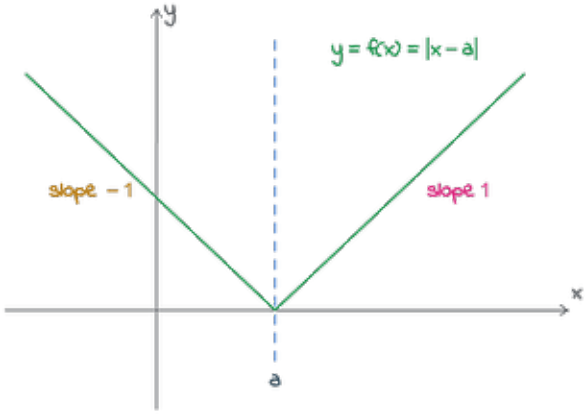
Left → $x < a \to |x - a| = -(x - a)$

$$\frac{-(x-a)}{x-a}$$

$= -1$

IDEA 2: THE TIP OF THE V, CONTINUOUS BUT NO SLOPE

|x − a| at its tip



$1 \neq -1 \rightarrow$ no limit $\rightarrow$ not differentiable at a

$\lim_{x \to a} |x - a| = 0 = f(a) \rightarrow$ continuous at a

Continuous $\Rightarrow$ differentiable ✗

Not continuous $\Rightarrow$ not differentiable ✓

a not in domain $\rightarrow$ no $f(a) \rightarrow$ no $f(a+h) - f(a) \rightarrow$ no $f'(a)$

WHERE THE SLOPE FAILS: CORNER, CUSP, VERTICAL TANGENT

No break, still no slope

No break, still no slope



A ruler rocks $\rightarrow$ corner

Sharp point $\rightarrow$ cusp

Stands straight up $\rightarrow$ vertical tangent

Tangent exists, but no slope

$$\frac{\sqrt[3]{h} - \sqrt[3]{0}}{h} = \frac{\sqrt[3]{h}}{(\sqrt[3]{h})^3} = \frac{1}{(\sqrt[3]{h})^2} \rightarrow +\infty$$

Smooth look? Not enough $\rightarrow$ real limit

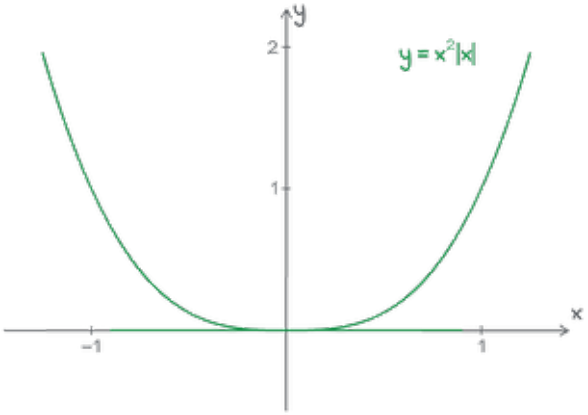
IDEA 3: CHECK, DON'T GUESS

Does |x| always make a corner?

Check, don't guess

$x = a + h \rightarrow x \rightarrow a^+ \Leftrightarrow h \rightarrow 0^+$

$x \rightarrow a^- \Leftrightarrow h \rightarrow 0^-$



$f(x) = x^2|x| \rightarrow f(0) = 0$

$$\frac{f(0+h) - f(0)}{h}$$

$$= \frac{h \cdot h|h|}{h} = h|h|$$

$h \rightarrow 0^+ \rightarrow h \cdot h = h^2 \rightarrow 0$

$h \rightarrow 0^- \rightarrow h(-h) = -h^2 \rightarrow 0$

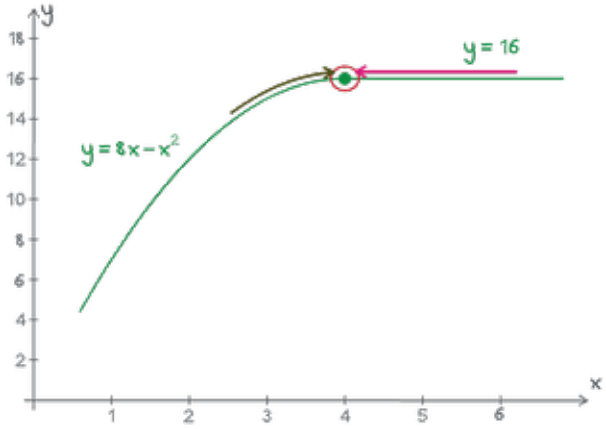
Both $\rightarrow 0 \rightarrow$ differentiable at 0

IDEA 4: AT A JOINT, HEIGHT FIRST

At a joint: height first

Joint $\rightarrow$ own piece each side, same $f(c)$

Height first $\rightarrow$ then each side



$f(x) = 8x - x^2$ $(x < 4)$, 16 $(x \geq 4)$

$f(4) \to 16$

$h < 0 \to f(4+h) - f(4) = 8(4+h) - (4+h)^2 - 16$

$= 32 + 8h - 16 - 8h - h^2 - 16 = -h^2$

$\frac{-h \cdot h}{h} = -h \to 0$

$h > 0 \to 16 - 16 = 0$

$\frac{0}{h} = 0$ (top exactly 0)

Left 0, right 0 → differentiable at 4

PROBLEM 1 OF 3

Where is f not differentiable?

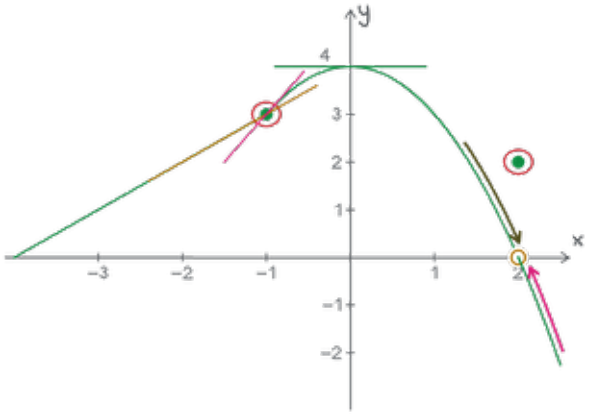
The graph of f is shown for $-4 < x < 2.5$. At which values of x is f not differentiable?

(A) $x = -1$ only

(B) $x = 2$ only

(C) $x = -1$ and $x = 2$ only

(D) $x = -1$, $x = 0$, and $x = 2$



Solution Two questions at each suspect

Plan → −1, 2, 0 → a break? one slope?

At −1 → meet at 3 → no break

Left → slope 1 → right → steeper

Two slopes → a corner → no $f'(-1)$

At 2 → x → 2, $f(x) \to 0$ → limit 0

But $f(2) = 2$ → a break → not differentiable

At 0 → smooth top → $f'(0) = 0$ → fine

A peak? Fine. Pointy? No.

(A) misses 2 ✗

(B) misses −1 ✗

(D) adds 0 → a smooth top ✗

So (C) → −1, 2

PROBLEM 2 OF 3

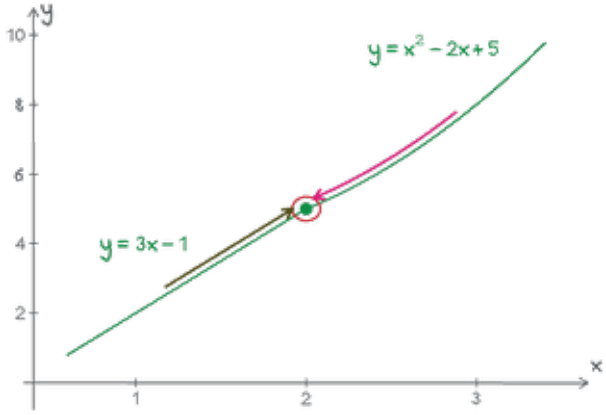
A joint at two

Let $f(x)=\begin{cases}3x-1, & x\le 2\\ x^2-2x+5, & x>2\end{cases}$

(a) Show that f is continuous at $x=2$.

(b) Find $\lim\limits_{h\to 0^-}\frac{f(2+h)-f(2)}{h}$ and $\lim\limits_{h\to 0^+}\frac{f(2+h)-f(2)}{h}$.

(c) Is f differentiable at $x=2$? Give a reason.



Solution (a) Continuity first

(a)

Plan → continuity → both quotients → compare

$f(2)=3(2)-1=5 \quad (x\le 2)$

$h<0 \to f(2+h)-f(2)=3(2+h)-1-5=6+3h-6=3h$

$h>0 \to f(2+h)-f(2)=(2+h)^2-2(2+h)+5-5$

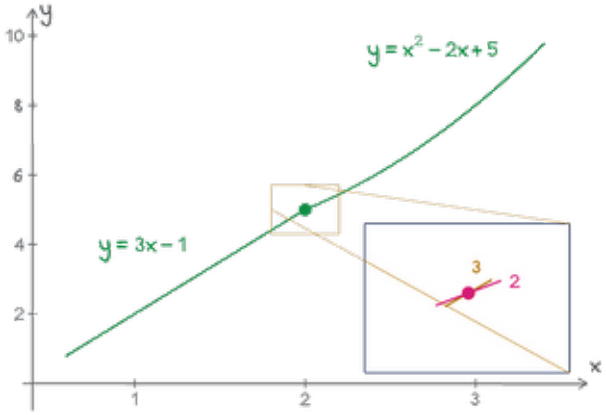
$=4+4h+h^2-4-2h=2h+h^2$

$\lim\limits_{h\to 0^-}3h=0, \quad \lim\limits_{h\to 0^+}(2h+h^2)=0$

$\lim\limits_{h\to 0}[f(2+h)-f(2)]=0$

$x=2 \to f$ is continuous

Solution (b)(c) Each side, then compare



(b)

Each side → its own quotient

$\lim\limits_{h\to 0^-}\frac{3h}{h}=3$

$\lim\limits_{h\to 0^+}\frac{2h+h^2}{h}=\lim\limits_{h\to 0^+}\frac{h(2+h)}{h}$

$(2+h)\to 2+0=2$

(c)

$3\neq 2 \to$ no limit

Not differentiable

Angle → corner

COMMON MISTAKE: CONTINUOUS, SO DIFFERENTIABLE?

Continuous → differentiable?

"f continuous at 2, so differentiable" X

Continuity needed, not enough

Reason: $3\neq 2$

PROBLEM 3 OF 3

A table and the IVT

g is differentiable for all real numbers. Selected values of g are in the table.

(a) Use $\lim_{h\to 0}[g(3+h)-g(3)]$ to show g is continuous at $x=3$.

(b) Must $g(c)=6$ for some c in $(1,3)$? Justify.

x	1	3	5
$g(x)$	9	4	8

Solution (a) The h form

(a)

Change in g (Δg) = quotient × h

$g(3+h)-g(3)=$

$\frac{g(3+h)-g(3)}{h}\cdot h \quad (h\neq 0)$

$\frac{g(3+h)-g(3)}{h}\to g'(3)$

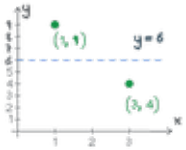
$h\to 0,\ g'(3)\cdot 0=0$

$\lim_{h\to 0}\Delta g=0$

g continuous at 3

Solution (b) Reason before the theorem

x	1	3	5
$g(x)$	9	4	8



(b)

g differentiable (given),

so g continuous on [1, 3].

$g(3)=4<6<9=g(1)$

By the IVT → c in (1, 3), g(c) = 6. Yes.

→ g(x) = 6 solvable

Reason first

WRAP-UP

What a real limit forces

Limit exists → top → 0 → continuous

Differentiable ⇒ continuous ✓

1 ≠ −1 at the tip of the V

Continuous ⇒ differentiable ✗

Not continuous ⇒ not differentiable ✓

corner, cusp, vertical tangent

Joint → height first → each side

|x| → check, don't guess

Workbook → video

Next → 2.5, the power rule

Next: the workbook for this topic (link in the video description). Stuck? Ask in the comments.