Lecture Notes · Unit 2 · Lecture 2 AP CED 2.2

The derivative function and tangent lines

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INTRO: A NEW LIMIT FOR EVERY INSTANT?

The Derivative Function AP Calculus 2.2

one point → one number: $f'(a)$

Run: 100 m, 20 s; $f(t)$ = distance by t

7 s → $f'(7)$, one limit. 10 s? 15 s?

A new limit for every instant!

FROM F'(A) TO F'(X): PUT X THERE

One limit with x

$$f'(7) = \lim_{h \to 0} \frac{f(7+h) - f(7)}{h}$$

$$f'(10) = \lim_{h \to 0} \frac{f(10+h) - f(10)}{h}$$

$$f'(15) = \lim_{h \to 0} \frac{f(15+h) - f(15)}{h}$$

⋮

purple → x

The instantaneous rate of change at x

x where limit exists ⇒ a function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f'(x)$ = the derivative of f

$f'(a)$ → a number; $f'(x)$ → a function

Bottom: $(x+h) - x = h$

Top: $h \to 0 \to f(x) - f(x)$ form ⇒ $f'(x)$

Runner: $f'(t)$ = speed at every instant

F'(X) FROM ONE LIMIT

$f'(x)$ from one limit

$f(x) = x^2 - 4x + 5 \to f'(x)$

In the limit: only h moves

$f(x+h) \to (x+h)^2 - 4(x+h) + 5$

$= x^2 + 2xh + h^2 - 4x - 4h + 5$

$-f(x) = -(x^2 - 4x + 5)$

$f(x+h) - f(x) = 2xh + h^2 - 4h$

$$f'(x) = \lim_{h \to 0} \frac{2xh + h^2 - 4h}{h}$$

$$= \lim_{h \to 0} \frac{h(2x + h - 4)}{h} \quad (h \neq 0)$$

$$= \lim_{h \to 0} (2x + h - 4) = 2x + 0 - 4 = 2x - 4$$

$f'(3) = 2(3) - 4 = 2 \to$ number

$f'(x) \to$ derivative

THE SAME LIMIT: 1/X AND THE SQUARE ROOT OF X

The same limit: $\frac{1}{x}$ and $\sqrt{x}$

$f(x) = \frac{1}{x}, x \neq 0$

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{x(x+h)h}$$

$$= \lim_{h \to 0} \frac{-h}{x(x+h)h}$$

$$= \frac{-1}{x(x+0)} = -\frac{1}{x^2}, x \neq 0$$

$f(x) = \sqrt{x}$

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$\times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}, \; x > 0$$

At $x = 0$, $2\sqrt{0} = 0 \rightarrow$ in 2.4

Again in 2.5

OTHER NAMES: Y', DY/DX, D/DX

Other names: y', $\frac{dy}{dx}$, $\frac{d}{dx}$

$f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx} f(x) \; (y = f(x))$

Average $\frac{\Delta y}{\Delta x} \xrightarrow{\Delta x \to 0} \frac{dy}{dx}$

one symbol, not a fraction → no canceling

$\frac{d}{dx}$ = operator → derivative

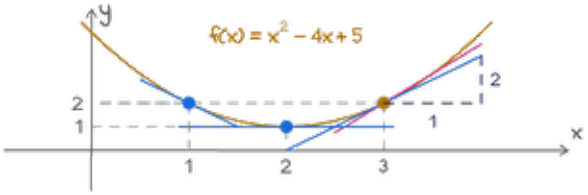
$\frac{d}{dx}\left(x^2 - 4x + 5\right) = 2x - 4$

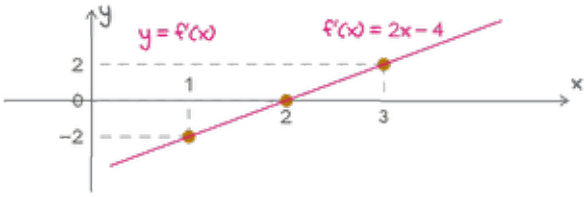
$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

$y = x^2 - 4x + 5, \; \left.\frac{dy}{dx}\right|_{x=3} = 2(3) - 4 = 2$

F'(A) IS THE SLOPE OF THE TANGENT

$f'(a)$ = slope of the tangent





$f'(a)$ = tangent slope at a

Tiny secant ($h \to 0$) → tangent

$x = 3$, right 1, up 2 → slope $\frac{2}{1} = 2 = f'(3)$

$x = 1$ → slope -2; $x = 2$ → slope 0

x	1	2	3
slope of f	−2	0	2

Slope of f → height on f', same x

One line: $f'(x) = 2x - 4$

Slope of f = height of f'

THE TANGENT LINE: POINT FIRST, THEN SLOPE

The tangent line

$y = f(a) + f'(a)(x - a)$

A slope alone → many parallel lines

point $(a, f(a))$ → slope $f'(a)$

$f(3) = 3^2 - 4(3) + 5 = 2 \rightarrow (3, 2)$

$y = 2 + f'(3)(x - 3) = 2 + 2(x - 3)$

Slope spot: the number 2, not $2x - 4$

$x = a \rightarrow f'(a)$; slope → $f'(x)$ = slope

PROBLEM 1 OF 3

f'(2) by the definition, then the tangent line

Let $f(x) = x^3 - 3x$.

(a) Use the definition of the derivative to find $f'(2)$.

(b) Write an equation of the line tangent to the graph of f at $x = 2$.

Definition

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Solution (a) The point, then f(2+h)

(a)

Plan: x = 2 → slope; point (2, f(2)) first

$$f(2) = 2^3 - 3(2) = 8 - 6 = 2$$

$$f(2+h) = (2+h)^3 - 3(2+h)$$

$$(2+h)^2 = (2+h)(2+h) = 4 + 4h + h^2$$

$$(2+h)^3 = (4 + 4h + h^2)(2+h)$$

$$= 8 + 8h + 2h^2 + 4h + 4h^2 + h^3$$

$$= 8 + 12h + 6h^2 + h^3$$

$$-3(2+h) = -6 - 3h$$

Combine the two expansions

$$f(2+h) = 2 + 9h + 6h^2 + h^3$$

Your turn: the limit, then watch

Solution (a) The limit

Definition

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

(a)

Plan: set it up → factor out h

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{(2 + 9h + 6h^2 + h^3) - 2}{h}$$

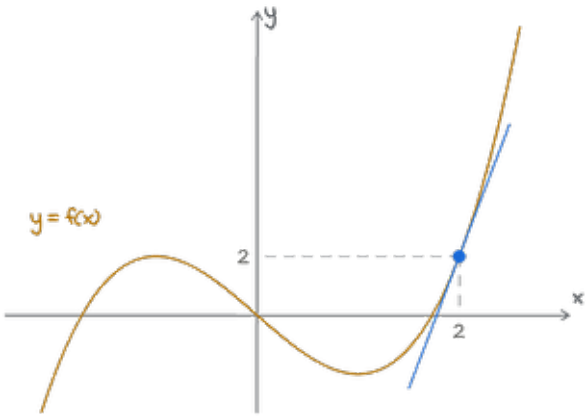
$$= \lim_{h \to 0} \frac{9h + 6h^2 + h^3}{h}$$

$$= \lim_{h \to 0} \frac{h(9 + 6h + h^2)}{h} \quad (h \neq 0)$$

$$= \lim_{h \to 0} (9 + 6h + h^2) = 9 + 6(0) + 0^2 = 9$$

Your turn: part (b), write the line

Solution (b) The tangent line



(b)

Point (2, f(2)) = (2, 2), slope f'(2) = 9

$$y = f(2) + f'(2)(x - 2)$$

$$y = 2 + 9(x - 2)$$

Already complete → no simplifying

Workbook: a tangent line at a given x

COMMON MISTAKE: THE LINE MISSES THE POINT

Student: $y = 9(x-2)$

$x = 2 \to y = 0$, not 2

Start at $(2,2)$ → walk with slope 9

$y = 2 + 9(x-2)$

$x = 2 \to y = 2$, on the curve

Point $(2,9)$; 7 = slope, not height

(b)

$(2,9)$, slope 7 → $y = 9 + 7(x-2)$

Workbook: a given slope, line by line

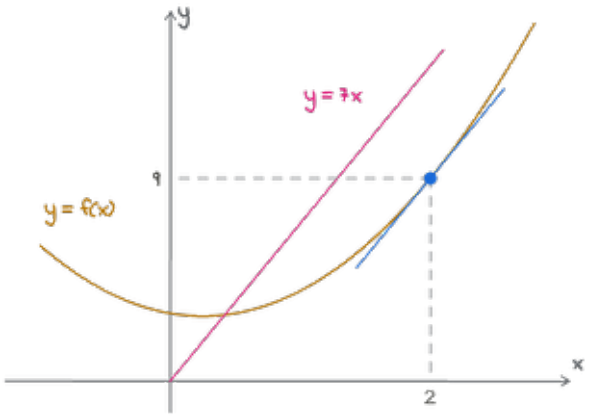
PROBLEM 2 OF 3

A parallel tangent

Let $f(x) = 2x^2 - x + 3$. By the definition, $f'(x) = 4x - 1$.

(a) Find the point where the tangent is parallel to $y = 7x$.

(b) Write an equation of that tangent line.



Solution Slope given: solve f'(x) = 7, then the point

(a)

Plan: $f'(x) = 7$ → point → line

Parallel → same slope, 7

$4x - 1 = 7$

$4x = 7 + 1 = 8 \Rightarrow x = 2$

$f(2) = 2(2)^2 - 2 + 3 = 8 - 2 + 3 = 9$

PROBLEM 3 OF 3

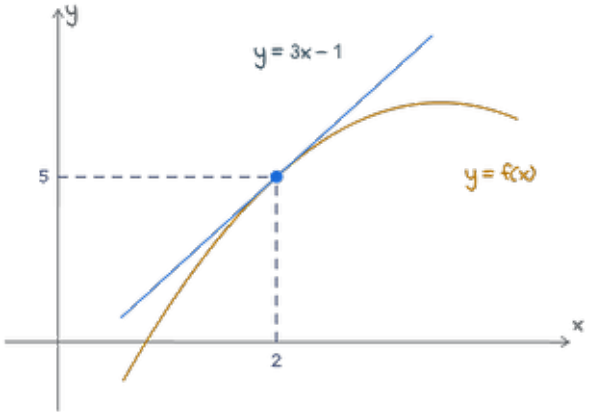
Read the tangent line

The line $y = 3x - 1$ is tangent to the graph of f at $x = 2$.

(a) Find $f(2)$.

(b) Find $f'(2)$.

(c) Find $\lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$.



Solution Same height, same slope

Plan: $x = 2$ → same height, same slope

(a)

$f(2) = 3(2) - 1 = 5$

(b)

$f'(2)$ = tangent slope = 3

5 = height, 3 = slope

(c)

$\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = f'(2) = 3$

workbook: read a tangent line

WRAP-UP



A new limit for every instant?

No → one limit with x → the function f'(x)

f'(3) → a number. f'(x) → the derivative

Slope of f = height of f'

Slope given → solve f'(x) = m first

Tangent line given → its height f(a), its slope f'(a)

Tangent line: point, then slope (a number)

Next → 2.3 → estimate f' from a table or graph

Next: the workbook for this topic (link in the video description). Stuck? Ask in the comments.