**Lecture Notes · Unit 2 · Lecture 1** AP CED 2.1

# The rate of change at an instant

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## SAM'S 100 M: AVERAGE VS. INSTANT

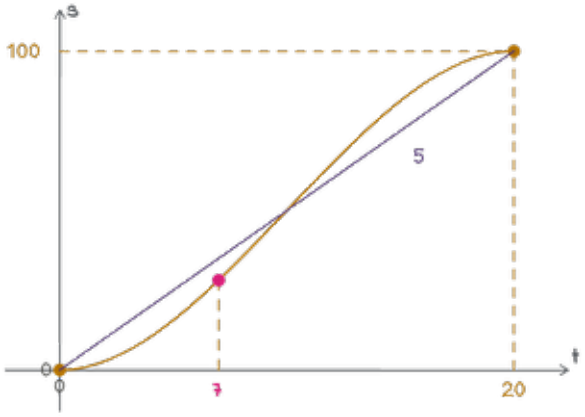
### Average vs. Instant AP Calculus 2.1

Sam: 100 m in 20 s. How fast?

$\frac{100}{20} = 5$ m/s

At $t = 7$ → still 5 m/s?

Today: the rate at one instant, exactly




Sam is not a treadmill.

$\frac{100-0}{20-0} = 5$

Slope of 2 points = average rate of change

$t = 7$ → the instant

## ZOOMING IN AT SEVEN SECONDS

### The instant $t = 7$

Freeze at $t = 7$ → no time → $\frac{0}{0}$?

Instantaneous velocity

instant → change in time $\Delta t \to 0$

$$\frac{\text{distance change } \Delta s}{\Delta t}$$

Zoom in at 7 → 7 + a neighbor → slope

Slide in: $\Delta t \to 0$, $\Delta s \to 0$

$\Delta t \neq 0$ → where slopes head

Slope: 7 and right next to it

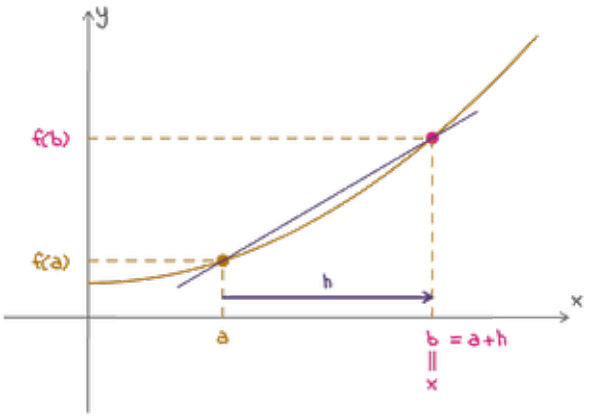
next to → limit; slope → avg. rate

Instant. rate = limit of avg. rates

Secant → tangent: slope = velocity at 7

## IN MATH: ONE SLOPE, THREE WAYS

### From Sam to any f




Average rate of change = 2-point slope

$\frac{f(b)-f(a)}{b-a}$

The step $h \to b = a+h$

$= \frac{f(a+h)-f(a)}{(a+h)-a}$

$(a+h)-a = h \to \frac{f(a+h)-f(a)}{h}$

b is also $x \to \frac{f(x)-f(a)}{x-a}$

h, x forms: difference quotients

## THREE FORMS, THREE ANSWERS?

$f(x) = \frac{x^2}{4} + 1 \quad a = 2 \quad b = 4$

$f(2) = \frac{2^2}{4} + 1 = 2$

$f(4) = \frac{4^2}{4} + 1 = 5$

$\frac{f(4) - f(2)}{4 - 2} = \frac{5 - 2}{2} = \frac{3}{2}$

$h = 2, x = 4 \to$ all $\frac{3}{2}$

## SQUARE MINUS TRIANGLE

$\frac{f(\square) - f(\triangle)}{\square - \triangle}$

Bottom = $\square - \triangle$ simplified → same value

$\frac{f(a + 3h) - f(a - h)}{4h}$

$(a + 3h) - (a - h) = 4h$ ✓

ours: $b$, $a + h$, $x$ in $\square$; $a$ in $\triangle$ ✓

## THE INSTANTANEOUS RATE: F'(A)

### The rate at an instant

At a, instant. rate = limit of avg. rates

$b \to a \to$ the step $h \to 0$, never 0

$b = x = a + h$, so $x \to a \iff h \to 0$

$\lim_{b \to a} \frac{f(b) - f(a)}{b - a}$

$= \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$

$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

Limit exists → $f'(a)$

$f'(a)$ = instant. rate at a = derivative at a

Units: f per unit x (m/s)

## READING A LIMIT: WHICH POINT?

### Given a limit: which point?

Given a limit → which point?

$\frac{f(\square) - f(\triangle)}{\square - \triangle}$ → bottom simplified

Limit: top → $f(a) - f(a) \to f'(a)$; $f(b) - f(b) \to f'(b)$

Try $\lim_{h \to 0} \frac{f(2 + h) - f(2)}{h}$

$(2 + h) - 2 = h$ ✓

$h \to 0 \to$ top → $f(2) - f(2) \to f'(2)$

Not $f'(0)$ → read the top

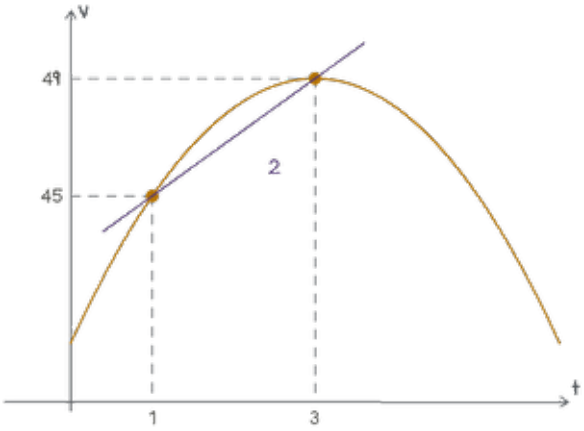
## PROBLEM 1 OF 2

Average and instant

The volume of water in a tank is $V(t) = 40 + 6t - t^2$ liters, $t$ in minutes, $0 \le t \le 6$.

(a) Find the average rate of change of $V$ over $1 \le t \le 3$, with units.

(b) Write a limit for the instantaneous rate of change at $t = 1$. Evaluate it, with units.




**Solution** (a) Two times → slope

(a)

Plan: two times → slope

$\frac{V(3) - V(1)}{3 - 1}$

$V(3) = 40 + 6(3) - 3^2 = 40 + 18 - 9 = 49$

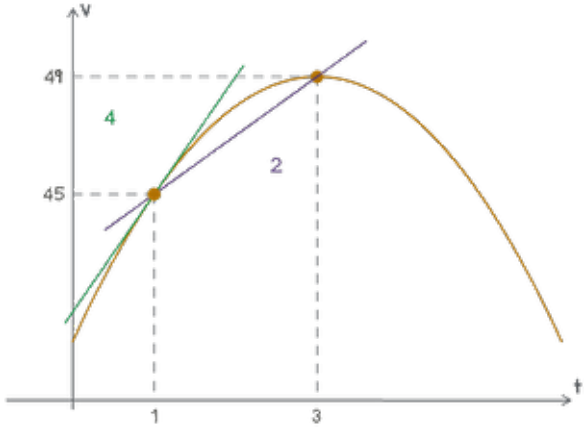
$V(1) = 40 + 6(1) - 1^2 = 45$

$= \frac{49 - 45}{2} = \frac{4}{2} = 2$

2 L/min, on average

Your turn: the limit for (b)

**Solution** (b) One instant → a limit




(b)

Plan: limit of avg. rates

$$\frac{V(1+h) - V(1)}{(1+h) - 1} = \frac{V(1+h) - V(1)}{h}$$

$V(1+h) = 40 + 6(1+h) - (1+h)^2$

$= 40 + 6 + 6h - (1 + 2h + h^2)$

$= 40 + 6 + 6h - 1 - 2h - h^2$

$= 45 + 4h - h^2$

$(45 + 4h - h^2) - 45 = 4h - h^2$

$$\lim_{h \to 0} \frac{V(1+h) - V(1)}{h} = \lim_{h \to 0} \frac{h(4-h)}{h}$$

$h \neq 0$ → cancel $h$

$= \lim_{h \to 0} (4 - h) = 4 - 0 = 4$

$V'(1) = 4$ liters per minute

## COMMON SLIP: REUSING THE AVERAGE

Common slip: reuse (a)?

(b) looks like (a) → reuse 2?

2 → from two times → average

one time, $t = 1$ → limit → 4

Avg. 2, instant 4

workbook → avg. + instant

## PROBLEM 2 OF 2

Four look-alikes

Let $f(x) = 3x^2 + 2x$. Which of the following is the instantaneous rate of change of $f$ at $x = -1$?

(A) $\lim_{h \to 0} \frac{f(-1+h) - f(-1)}{-1+h}$

(B) $\lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$

(C) $\lim_{h \to 0} \frac{f(h) - f(-1)}{h}$

(D) $\lim_{x \to -1} \frac{f(x) - f(-1)}{-1-x}$

Instant at a

$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Bottom

□ − △ simplified

**Solution** Square minus triangle first

Plan: □ − △ first, bottom ≠ → out

(A) □ − △ = (−1 + h) − (−1) = h ≠ −1 + h ✗

(C) □ − △ = h − (−1) = h + 1 ≠ h ✗

(D) □ − △ = x + 1 ≠ −1 − x ✗

So (B)

(B) $(-1+h)-(-1)=h$ ✓ → the h form at $a=-1$

$h \to 0$ → top → $f(-1)-f(-1)$ → $f'(-1)$, not $f'(0)$

(D): all the right numbers → fine?

$-1-x=-(x+1)$ → $-f'(-1)$ → sign flipped

workbook → □ − △, then the top

**WRAP-UP**

## Back to Sam

No: green tangent steeper than purple

Instant. rate = limit of average rates of change

Limit exists → one number, $f'(a)$

Bottom: □ − △ simplified

Read the top → the point

Units of f per unit of x

Next → 2.2 → $f'(x)$ = slope at every point

Next: the workbook for this topic (link in the video description). Stuck? Ask in the comments.