Lecture Notes · Unit 1 · Lecture 17 AP CED 1.16

IVT

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INTRO: AN EQUATION, NO SOLVING

The Intermediate Value Theorem AP Calculus 1.16

Last: $x \to \pm\infty \to$ today: x in $[a, b]$

Today an equation → no factoring

A root, no solving → 3 sentences

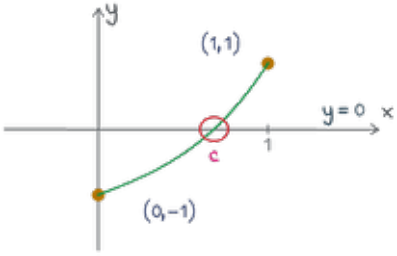
IDEA 1: DOES THE EQUATION HAVE A SOLUTION?

Key ideas

$x^3 + x - 1 = 0$ → a solution?

1. An equation: is there a solution?

$f(x) = x^3 + x - 1$ $f(0) = -1$, $f(1) = 1$



must cross?

No break: below → above → crosses $y = 0$

$f(x) = 0$ → root c in $(0, 1)$ → at least one

1. f continuous on $[a, b]$

k between $f(a)$, $f(b)$ $k = 0$ here

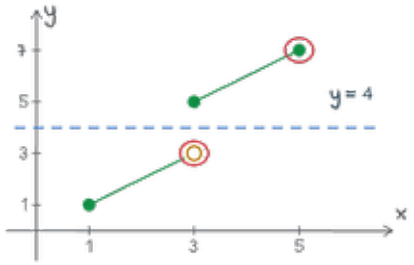
$f(x) = k$ → root c in (a, b)

Intermediate Value Theorem → IVT

IDEA 2: WHY CONTINUOUS? A BREAK

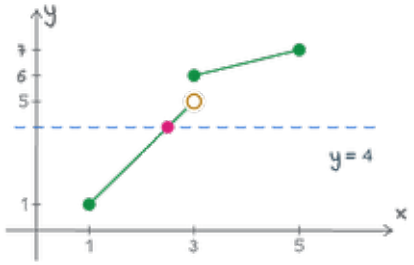
Key ideas

2. Why continuous? A break



$1 < 4 < 7$, f never $= 4$ → the jump skips it

Break on $[1, 5]$ → no IVT guarantee



No promise ≠ no c (just no promise)

Break, or k outside $f(a)$, $f(b)$ → no promise

IDEA 3: THREE SENTENCES

Key ideas

3. Three sentences, the reason first

Condition first → continuity

e.g. $f(x) = x^3 - 2x^2 + 3$ on $[1, 3]$; $f(c) = 7$?

Polynomial → continuous on $[1, 3]$

or → "f is continuous" is given

Numbers

$f(1) = 1^3 - 2(1)^2 + 3 = 2$

$f(3) = 3^3 - 2(3)^2 + 3 = 12$

Yes: $f(1) = 2 < 7 < 12 = f(3)$

By IVT $f(c) = 7$ for some c in $(1, 3)$ (AP)

$f(x) = 7$: a root in $(1, 3)$

Three: reason · numbers · conclusion

IDEA 4: MAKE IT H(X) = 0, FIND A ROOT C

Key ideas

4. Make it $h(x) = 0$, find a root c

$h(x) = 0 \rightarrow$ a root c

$f(x) = k \Leftrightarrow f(x) - k = 0$ like idea 1

$\cos x = x$, $0 < x < \frac{\pi}{2}$?

$\cos x = x \Leftrightarrow \cos x - x = 0$

$h(x) = \cos x - x \rightarrow h(c) = 0$

$\cos x$, x continuous $\rightarrow$ h continuous on $\left[0, \frac{\pi}{2}\right]$

$h(0) = \cos 0 - 0 = 1 > 0$

$h\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} - \frac{\pi}{2} = -\frac{\pi}{2} < 0$

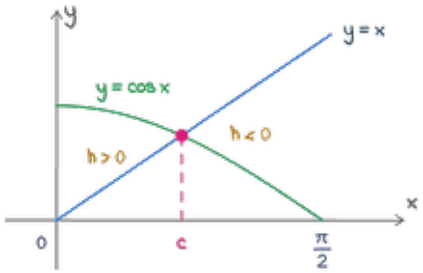
$h\left(\frac{\pi}{2}\right) < 0 < h(0)$ smaller first

IVT $\rightarrow h(c) = 0$, root c in $\left(0, \frac{\pi}{2}\right)$

At that root $\rightarrow \cos c = c$

IDEA 4: THE PICTURE, F(X) = G(X)

Key ideas



In general $f(x) = g(x) \rightarrow$ graphs meet

Move $g(x) \rightarrow f(x) - g(x) = 0$

$h(x) = f(x) - g(x) \rightarrow c =$ a root of $h(x) = 0$

$f > g \rightarrow h > 0$

$f < g \rightarrow h < 0$

f, g continuous $\rightarrow h$ too; sign flips $\rightarrow$ IVT, $k = 0$

$h(c) = 0 \rightarrow f(c) = g(c)$

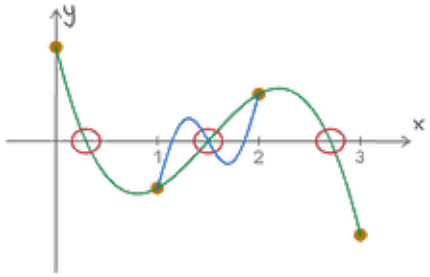
IDEA 5: WHAT THE IVT DOESN'T SAY

Key ideas

5. What the IVT doesn't say

f continuous: $[0, 3]$

$f(0) = 2$, $f(1) = -1$, $f(2) = 1$, $f(3) = -2$



$\rightarrow$ at least 3 roots, maybe more

$-1 < 0 < 2 \rightarrow$ at least one root $\neq$ exact count

No exact place $\rightarrow$ write "at least one"

Break $\rightarrow$ no IVT · maybe a root

PROBLEM 1 OF 3

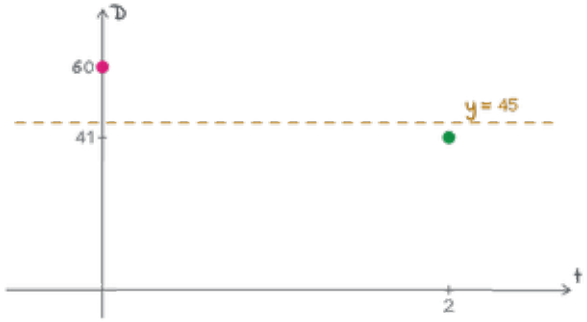
A table in context

The depth of water $D(t)$, in centimeters, is modeled by a continuous function of time t, in hours. Selected values of $D(t)$ are given in the table. Must there be a time t, with $0 < t < 8$, at which $D(t) = 45$? Justify your answer.

t	$D(t)$
0	60
2	41
3	30
5	52
8	70

Solution Find where the depth crosses 45 cm

t	0	2	3	5	8
D(t)	60	41	30	52	70



Plan: find two depths around 45

D continuous on [0, 2]

D(2) = 41 < 45 < 60 = D(0)

D(t) = 45 for some t in (0, 2), inside (0, 8)

Workbook → inside the table

Common mistake

t	D(t)
0	60
2	41
3	30
5	52
8	70

✗ Common mistake

Yes, by the IVT.

Reason? Numbers?

Continuity?

✓ Do this

D continuous on [0, 2]

D(2) = 41 < 45 < 60 = D(0)

D(t) = 45 for some t in (0, 2), inside (0, 8)

PROBLEM 2 OF 3

Is there c with f(c) = c?

The function f is continuous on $[0, 5]$. Selected values of f are given in the table.

Must there be a c, with $0 < c < 5$, such that $f(c) = c$? Justify your answer.

x	f(x)
0	2
1	3
3	1
5	4

Solution Turn f(c) = c into a root of the difference

Plan: f(c) = c → a difference = 0

f(c) = c ⇔ f(c) − c = 0

g(x) = f(x) − x → a root of g(x) = 0

g(1) = f(1) − 1 = 3 − 1 = 2

g(3) = f(3) − 3 = 1 − 3 = −2

f continuous (given), x polynomial → g continuous on [1, 3]

g(3) = −2 < 0 < 2 = g(1)

By the IVT → g(c) = 0 for some c in (1, 3)

f(c) − c = 0 → f(c) = c

Workbook → name the difference first

PROBLEM 3 OF 3

A break inside the interval

Let $f(x)=\dfrac{x+2}{x-2}$, so $f(1)=-3$ and $f(3)=5$. Which of the following is true?

(A) The IVT guarantees a solution of $f(x)=0$ in $(1,3)$.

(B) $f(c)=0$ for some c in $(1,3)$.

(C) The IVT cannot be applied to f on $[1,3]$.

(D) $f(c)=1$ for some c in $(1,3)$.

Solution Check the condition first

Plan: check continuity on [1, 3] first

x − 2 = 0 at x = 2

1 < 2 < 3 → f(2) undefined

Break in [1, 3] → no IVT

(A) IVT guarantee? X

(B) x + 2 = 0 → x = −2 outside (1, 3) X

(D) f(x) = 1 → x + 2 = x − 2 → 2 = −2 X

(C) Break at 2 → no IVT

workbook → condition first

WRAP-UP

A solution? Yes

Yes → f(c) = 0 for some c in (0, 1)

Continuous + k between → f(x) = k has a root

write 3 parts → reason, numbers, conclusion

Break, or k outside → IVT says nothing

Unit 1 → Unit 2: rate at an instant

Next: the workbook for this topic (link in the video description). Stuck? Ask in the comments.