Lecture Notes · Unit 1 · Lecture 16 AP CED 1.15

At infinity

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INTRO: BOTH ENDS

Limits at Infinity AP Calculus 1.15

Last: $x \to a$, $f \to \pm\infty$ → today: $x \to \pm\infty$

Far out → where does $f(x)$ settle?

1 function → 2 ends → 2 asymptotes?

Left end + / − → hidden minus → catch it

IDEAS 1–2: BOTH ENDS, EACH ON ITS OWN

Key ideas

1. Both ends

$x \to \infty$ → far right, $x \to -\infty$ → far left (1.3)

$\frac{5}{x}$ $\lim_{x\to\infty} \frac{5}{x} = 0$, $\lim_{x\to-\infty} \frac{5}{x} = 0$

Think $\frac{5}{\infty} \to 0$

Any $\frac{c}{x}$, $\frac{c}{x^2}$, $\frac{c}{x^3}$, ... → 0 (higher, faster)

$\sin x$: −1 to 1; $x \to \infty$ → $\frac{\sin x}{x} \to 0$ (1.8)

f at both ends → end behavior

2. Each end on its own

e.g. $f(x) = \frac{6}{2+e^x}$

$x \to \infty$ → e^x grows → bottom grows

Think $\frac{6}{\infty} \to 0$ → $\lim_{x\to\infty} f(x) = 0$

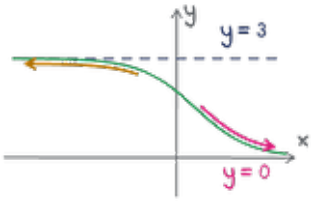
$x \to -\infty$: $t = -x \to \infty$, $e^x = e^{-t} = \frac{1}{e^t} \to 0$

$\frac{6}{2+0} = 3$ → $\lim_{x\to-\infty} f(x) = 3$

AP: no ∞ in the arithmetic → write $e^x \to 0$

IDEA 2: THE GRAPH, HORIZONTAL ASYMPTOTES

Key ideas



Real limit L at an end → line $y = L$

→ a horizontal asymptote

2 ends → $y = 0$, $y = 3$

Vertical $x \to a$; horizontal $x \to \pm\infty$

IDEA 3: BIG OVER BIG, DIVIDE BY THE HIGHEST POWER

Key ideas

3. Big over big

e.g. $\frac{6x^3 - x}{2x^3 + 7}$ as $x \to \infty$

Both unbounded → no value yet

Divide every term by x^3 (highest in bottom)

$$= \lim_{x\to\infty} \frac{\frac{6x^3}{x^3} - \frac{x}{x^3}}{\frac{2x^3}{x^3} + \frac{7}{x^3}}$$

$$= \lim_{x\to\infty} \frac{6 - \frac{1}{x^2}}{2 + \frac{7}{x^3}}$$

Top → 6, bottom → $2 \neq 0$ → quotient law

$= \frac{6-0}{2+0} = 3$

IDEA 4: $X \to -\infty$, LET $T = -X$

Key ideas

x^3 on both → leading ratio → $\frac{6}{2}$

Top degree lower → $\frac{x}{x^2} = \frac{1}{x} \to 0$

Top higher → $\frac{x^2}{x} = x \to \infty$

Compare degrees → show division

4. Negative infinity

$x \to -\infty \Rightarrow t = -x$

$x \to -\infty \to t = -x \to \infty$, and $t > 0$

$\lim_{x \to -\infty} f(x) = \lim_{t \to \infty} f(-t)$

e.g. $\lim_{x \to -\infty} \frac{\sqrt{x^2+1}}{x}$

$(-t)^2 = t^2 \to \lim_{t \to \infty} \frac{\sqrt{t^2+1}}{-t}$

Divide by $t > 0$ $\lim_{t \to \infty} \left(-\sqrt{1+\frac{1}{t^2}}\right) = -1$

Right end → 1 → ends differ: 1, −1

IDEA 5: WHO GROWS FASTER, THE LADDER

Key ideas

5. Who grows faster? The ladder

Growth as $x \to \infty$, slow → fast

$\ln x < \sqrt{x} < x < x^2 < x^n < e^x$ $(n > 2$, large $x)$

$\frac{\text{fast}}{\text{slow}} \to \infty$ $\frac{\text{slow}}{\text{fast}} \to 0$

Same rung → the leading coefficients

$\lim_{x \to \infty} \frac{x^2}{e^x} = 0$ (slower on top)

$\lim_{x \to \infty} \frac{\ln x}{x} = 0$ (slower on top)

$\lim_{x \to \infty} \frac{e^x}{x^3}$ (faster on top) $= \infty$

IDEA 6: CAN A CURVE CROSS ITS ASYMPTOTE?

Key ideas

6. Can a curve cross its asymptote?

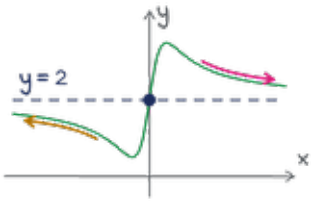
e.g. $f(x) = 2 + \frac{3x}{x^2+1}$

$x \to \pm\infty$: $\frac{3x}{x^2+1} = \frac{\frac{3}{x}}{1+\frac{1}{x^2}} \to \frac{0}{1} = 0$

$\lim_{x \to \infty} f(x) = 2$, $\lim_{x \to -\infty} f(x) = 2$

→ $y = 2$ is a horizontal asymptote

$f(0) = 2 + \frac{0}{0+1} = 2$



Crosses at $(0, 2)$, still an asymptote

ends only · may cross

PROBLEM 1 OF 3

A root, two ends

Let $f(x) = \frac{\sqrt{4x^2+7}}{x+3}$. A student says $\sqrt{4x^2} = 2x$, so $y = 2$ is the only horizontal asymptote.

(a) Find $\lim_{x \to \infty} f(x)$.

(b) Find $\lim_{x \to -\infty} f(x)$.

(c) Is the student right?

Solution Each end on its own, with the sign of x

Plan: each end on its own → left end: $t = -x$

(a)

$x \to \infty \to x > 0$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\sqrt{4+\frac{7}{x^2}}}{1+\frac{3}{x}}$$

$\frac{7}{x^2} \to 0, \frac{3}{x} \to 0 \to$ bottom $\to 1 \neq 0$

$$= \frac{\sqrt{4}}{1} = 2$$

Solution Each end on its own, with the sign of x

(b)

Let $t = -x \to t \to \infty, t > 0$

$$\lim_{x\to-\infty} f(x) = \lim_{t\to\infty} \frac{\sqrt{4t^2+7}}{-t+3}$$

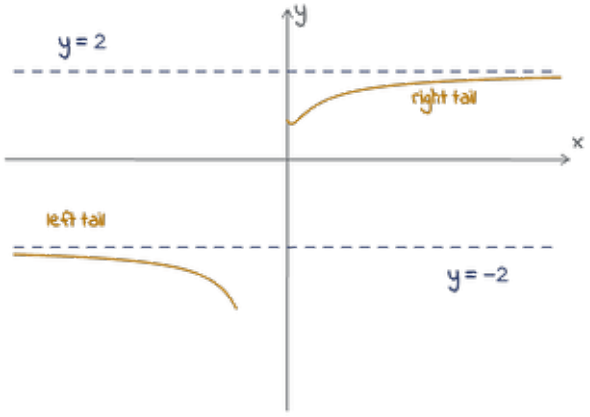
$\div t$ $\lim_{t\to\infty} \frac{\sqrt{4+\frac{7}{t^2}}}{-1+\frac{3}{t}}$

Bottom $\to -1 \neq 0$

$$= \frac{\sqrt{4}}{-1} = -2$$

workbook → each end separately

Solution Each end on its own, with the sign of x



Plan: both ends vs. the student

(c)

$\sqrt{4x^2} = 2|x|$, not $2x$

$y = 2$ and $y = -2$ → 2 asymptotes, not 1

$x \to +\infty$: 2; $x \to -\infty$: −2

workbook $\sqrt{x^2} = |x|$

Common mistake

X Common mistake

$\sqrt{4x^2} = 2x$

$\lim_{x\to-\infty} f(x) = 2$

✓ Do this

$t = -x > 0$

$\frac{\sqrt{4t^2+7}}{-t+3} \to -2$

Left end

PROBLEM 2 OF 3

Which one has y = 3?

Which of the following functions has the horizontal asymptote $y = 3$?

(A) $\dfrac{\sin 6x}{2x}$

(B) $\dfrac{3x^2}{x+4}$

(C) $\dfrac{3-6x}{2x+5}$

(D) $\dfrac{3e^x}{e^x+2}$

Solution Check the ends, not x → 0

Plan: the ends, $x \to \pm\infty$, not $x \to 0$

(A) → 3 near 0 · $-1 \le \sin 6x \le 1$ → ends: 0

$$\lim_{x\to\infty}\frac{3e^x}{e^x+2}=\lim_{x\to\infty}\frac{\frac{3e^x}{e^x}}{\frac{e^x}{e^x}+\frac{2}{e^x}}$$

$$=\lim_{x\to\infty}\frac{3}{1+\frac{2}{e^x}}\quad\left(\frac{2}{e^x}\to 0,\ \text{bottom}\to 1\right)$$

$$=\frac{3}{1}=3$$

(B) degree $2>1\to|y|\to\infty$ X

(C) same degree $\to\frac{-6}{2}=-3\neq 3$ X

So (D) $\to y=3$

workbook → the ends, not zero

PROBLEM 3 OF 3

In the long run

A cold drink sits in a room. Its temperature, in degrees Fahrenheit, after t minutes is

$$T(t)=\frac{70t+400}{t+10}.$$

Write a limit for the temperature in the long run, and find its value.

Solution The long run is $t\to\infty$

Plan: long run $\to t\to\infty$, not $t\to 0$

$$T(0)=\frac{400}{10}=40\ \text{(the start)}$$

$$\lim_{t\to\infty}T(t)=\lim_{t\to\infty}\frac{70t+400}{t+10}$$

$$=\lim_{t\to\infty}\frac{70+\frac{400}{t}}{1+\frac{10}{t}}$$

As $t\to\infty\to$ both terms $\to 0$

$1\neq 0$ (bottom) $\to$ quotient law

$$=\frac{70}{1}=70$$

Long run → T approaches 70°F

workbook → limit first, then value

WRAP-UP

f(x) far out

2 ends: finite L → y = L

Left end + ⌄ → t = −x first

Faster grower decides → polynomials: the degrees

e^x beats every power of x → e^x wins

Asymptote → the ends; the middle can cross

Long run → $t\to\infty$, not $t\to 0$

Next → 1.16 → the Intermediate Value Theorem

Next: the workbook for this topic (link in the video description). Stuck? Ask in the comments.