1.15 / Limits at Infinity and Horizontal Asymptotes

Watch the lesson ➤

Connecting limits at infinity and horizontal asymptotes

Learning goal
Find limits as $x \to \infty$ and $x \to -\infty$, use them to name horizontal asymptotes and end behavior, and compare how fast functions grow.

Key ideas

01 Far out, where does $f(x)$ head? $x \to \infty$ is the far right, $x \to -\infty$ the far left. $\lim_{x\to\infty} \frac{5}{x} = 0$, $\lim_{x\to-\infty} \frac{5}{x} = 0$; any $\frac{c}{x}, \frac{c}{x^2}, \frac{c}{x^3}, \ldots \to 0$. Where f heads is its **end behavior**.

02 Each end on its own: $f(x) = \frac{6}{2+e^x}$. As $x \to \infty$, e^x grows, so the bottom grows: think $\frac{6}{\infty} = 0$, $\lim_{x\to\infty} f(x) = 0$. As $x \to -\infty$, think $e^{-\infty} = \frac{1}{e^{\infty}} = 0$. AP: "$e^x \to 0$, so $\lim_{x\to-\infty} \frac{6}{2+e^x} = \frac{6}{2} = 3$." A real limit L at an end gives the line $y = L$, a horizontal **asymptote**: here $y = 0$ and $y = 3$.

03 Divide by the highest power: $\lim_{x\to\infty} \frac{6x^3 - x}{2x^3 + 7}$. Both grow without bound, so no value yet. Divide every term by x^3, the highest in the bottom (the same on the top and the bottom, so the same value): $\frac{6 - \frac{1}{x^2}}{2 + \frac{7}{x^3}}$. The top $\to 6$, the bottom $\to 2 \neq 0$: the limit is 3. Same degree: only the leading terms count. Lower on top: $\frac{x}{x^2} = \frac{1}{x} \to 0$. Higher on top: $\frac{x^2}{x} = x$, no real limit. The **degrees** decide.

04 $x \to -\infty$: let $t = -x$, so $t \to \infty$ and $t > 0$: $\lim_{x\to-\infty} f(x) = \lim_{t\to\infty} f(-t)$. E.g. $\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x} = \lim_{t\to\infty} \frac{\sqrt{t^2+1}}{-t} = \lim_{t\to\infty} \left(-\sqrt{1+\frac{1}{t^2}}\right) = -1$. No sign trap: $t > 0$, **always**. The right end gives 1: the ends can differ.

05 Who grows faster, slow to fast as $x \to \infty$: $\ln x < \sqrt{x} < x < x^2 < x^n < e^x$. Faster over slower: ∞; slower over faster: 0; the same rung: the **leading coefficients**. $\lim_{x\to\infty} \frac{x^2}{e^x} = 0$, $\lim_{x\to\infty} \frac{\ln x}{x} = 0$, $\lim_{x\to\infty} \frac{e^x}{x^3} = \infty$.

06 Crossing an asymptote: $f(x) = 2 + \frac{3x}{x^2+1}$. As $x \to \pm\infty$, $\frac{3x}{x^2+1} \to 0$ (lower on top), so $y = 2$ is a horizontal asymptote, yet $f(0) = 2$: it crosses in the middle and is still an **asymptote**. A horizontal asymptote is about the ends only. Ends can differ: $\frac{2e^x + 1}{e^x - 3} \to 2$ as $x \to \infty$; as $x \to -\infty$, $e^x \to 0$, so it heads to $\frac{0+1}{0-3} = -\frac{1}{3}$.

Remember

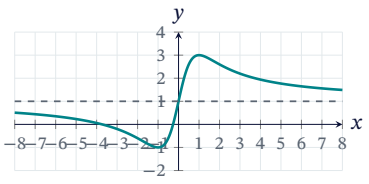
limit at infinity
The value $f(x)$ approaches as x grows without bound ($x \to \infty$) or decreases without bound ($x \to -\infty$).

end behavior
What $f(x)$ does at the far right and the far left of the graph.

horizontal asymptote
A line $y = L$, where L is a limit of $f(x)$ as $x \to \infty$ or as $x \to -\infty$.

leading term
The term with the highest power of x, such as $-4x^2$ in $5 - 4x^2$. It may not be written first.

The asymptote describes the ends; the graph may cross it



As $x \to \infty$ and as $x \to -\infty$, $f(x) \to 1$, so $y = 1$ is a horizontal asymptote. The graph crosses $y = 1$ at $x = 0$.

Class notes

1.15 / Write it clearly

What earns the point on the AP Exam, and the mistakes that lose it.

Writing checklist

Notation to use and to avoid

WRITE

Keep lim on every line until the last step: $\lim_{x\to\infty} \frac{8x-3}{4x+1} = \lim_{x\to\infty} \frac{8-\frac{3}{x}}{4+\frac{1}{x}} = \frac{8-0}{4+0} = 2$.

In context, set up the limit with $t \to \infty$ and say what the quantity approaches, with units: "$\lim_{t\to\infty} R(t) = 40$: the rate approaches 40 cubic meters per hour."

AVOID

Do not do arithmetic with ∞: never write $\frac{5}{\infty} = 0$, $\frac{\infty}{\infty} = 1$, or $\infty - \infty = 0$.

Do not say the quantity "is" the limit or "reaches a maximum of" the limit. Say that it approaches the limit.

Common mistakes

The mistake code is used in the answers and in My Mistakes

01 Find the leading terms, with their signs

Code L4

✗ $\lim_{x\to\infty} \frac{7x^2+1}{2x-x^2} = 7$, because the leading coefficients are 7 and 1.

✓ The leading term of the bottom is $-x^2$. Divide by x^2: $\lim_{x\to\infty} \frac{7+\frac{1}{x^2}}{\frac{2}{x}-1} = \frac{7}{-1} = -7$.

02 For x < 0, the root of x squared is -x

Code L13

✗ $\lim_{x\to-\infty} \frac{\sqrt{x^2+9}}{2x} = \frac{1}{2}$, because $\sqrt{x^2} = x$.

✓ Let $t = -x$, so $t \to \infty$ and $t > 0$: $\lim_{t\to\infty} \frac{\sqrt{t^2+9}}{-2t} = \lim_{t\to\infty} \frac{\sqrt{1+\frac{9}{t^2}}}{-2} = -\frac{1}{2}$.

03 The long run means t goes to infinity

Code L13

✗ "In the long run, the rate $R(t) = \frac{50t+8}{t+4}$ is $\lim_{t\to 0} R(t) = 2$."

✓ $\lim_{t\to\infty} \frac{50t+8}{t+4} = \lim_{t\to\infty} \frac{50+\frac{8}{t}}{1+\frac{4}{t}} = 50$. The rate approaches 50; $R(0) = 2$ is the starting rate.

Real AP questions on this topic Free on AP Central. Try them after the practice.

- 2025 AB FRQ 1(c) – a long-run rate written as a limit as t goes to infinity, then its value (same question as BC 1(c)); the most common error sent t to 0 instead of infinity

1.15 / Guided examples

Watch, pause, and write the reasoning. Full solutions are in the video.

Problem types

Read the problem, name the type, then use its method

Type	Method	Problems
Rational function as $x \to \pm\infty$: compare degrees	Divide the top and the bottom by the highest power of x in the bottom; each $\frac{c}{x^n} \to 0$. Top degree lower: the limit is 0. Same degree: the ratio of the leading coefficients, with their signs.	Practice 1
Top degree higher: unbounded end behavior and its sign	There is no horizontal asymptote. The quotient of the leading terms decides ∞ or $-\infty$; for $x \to -\infty$, remember that x is negative.	Practice 7
A root: $\sqrt{x^2} = \lvert x \rvert$, so the two ends can differ	At the right end, divide by x, since $\sqrt{x^2} = x$ for $x > 0$. At the left end, let $t = -x$: then $t \to \infty$ and $t > 0$. Find each end separately.	E01 · Practice 2
End behavior of exponential and logarithmic functions	As $x \to \infty$, e^x grows without bound: divide by e^x, then $e^{-x} \to 0$. As $x \to -\infty$, $e^x \to 0$. $\ln x$ grows without bound, slowly. Find each end separately.	
Compare growth rates with a limit	As $x \to \infty$, an exponential b^x $(b > 1)$ grows faster than any power of x, and any power x^p $(p > 0)$ grows faster than $\ln x$. Slower over faster approaches 0; divide by the fastest term.	Practice 3
Find every horizontal asymptote, or choose the function with $y = c$	Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Each real-number limit L gives the asymptote $y = L$. Look at $x \to \pm\infty$, not at $x \to 0$.	E02 · Practice 6
End behavior from a table; a graph can cross its asymptote	Read where the values head as x grows in each direction (a table only suggests). The asymptote describes only $x \to \pm\infty$: solve $f(x) = L$ to see where the graph crosses $y = L$.	Practice 4
A limit at infinity in context	Send the time to ∞, not to 0. Find the limit and say what the quantity approaches, with units.	E03 · Practice 5
A difference with a root: multiply by the conjugate	Both terms grow without bound, so rewrite. Multiply the top and the bottom by the conjugate, then divide by x (here $x > 0$, so $\sqrt{x^2} = x$).	

Example 01

Video · E01

Let $f(x) = \dfrac{\sqrt{4x^2+7}}{x+3}$. A student says $\sqrt{4x^2} = 2x$, so $y = 2$ is the only horizontal asymptote.
(a) Find $\lim_{x\to\infty} f(x)$.
(b) Find $\lim_{x\to-\infty} f(x)$.
(c) Is the student right?

> For $x < 0$, $\sqrt{x^2}$ is not x.

Example 02

Video · E02

Which function has the horizontal asymptote $y = 3$?

A. $\dfrac{\sin 6x}{2x}$

B. $\dfrac{3x^2}{x+4}$

C. $\dfrac{3-6x}{2x+5}$

D. $\dfrac{3e^x}{e^x+2}$

> Check $x \to \infty$ and $x \to -\infty$, not $x \to 0$.

Example 03

Video · E03

A cold drink sits in a room. Its temperature, in degrees Fahrenheit, after t minutes is
$T(t) = \dfrac{70t + 400}{t + 10}$.
Write a limit for the temperature in the long run, and find its value.

> The long run means $t \to \infty$, not $t \to 0$.

VIDEO COMPLETE Check your example answers at the end of this booklet.

1.15 / Example answers

Limits at Infinity and Horizontal Asymptotes

E01 (a) 2; (b) -2; (c) $\sqrt{4x^2} = 2|x|$, and for $x < 0$, $|x| = -x$. So the left end gives -2: there are two asymptotes, $y = 2$ and $y = -2$

E02 (D)

E03 $\lim_{t\to\infty} T(t) = 70$: in the long run the temperature approaches 70°F