Lecture Notes · Unit 1 · Lecture 15 AP CED 1.14

Infinite limits

Tap a blue play button next to a heading to open that part of the video. **▶ Watch the video**

INTRO: UP OR DOWN?

Infinite Limits and Vertical Asymptotes

AP Calculus 1.14

Near an asymptote → up, or down?

1.3 → $\frac{1}{x}$: $0^+ \uparrow \cdot 0^- \downarrow$

Signs decide → AP wording

IDEA 1: THE SIGNS PICK THE DIRECTION

Key ideas

1. The signs decide the direction

0^+ or 0^-

$f(x) = \frac{x-1}{(x-1)(x-3)}$

At 1 → $(x-1)$ cancels → $\frac{1}{x-3} \to -\frac{1}{2}$

Real limit, no $f(1)$ → hole, not an asymptote

At 3 → top → 2, bottom → 0 → no bound

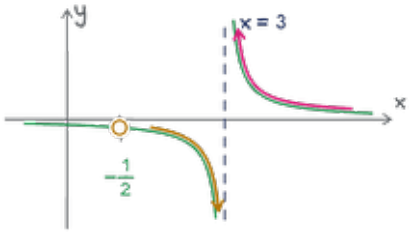
$x \to 3^-$ → $x-3<0, x-1>0$

bottom → 0^-

Think: $\frac{2}{0^-} \to -\infty \to \lim_{x \to 3^-} f(x) = -\infty$

$x \to 3^+$?

$x \to 3^+$ → bottom → 0^+ → $\lim_{x \to 3^+} f(x) = \infty$



IDEA 1: WRITE IT IN WORDS

Key ideas

AP: $x \to 3^-$, top → 2, bottom → 0 from below

$\lim_{x \to 3^-} f(x) = -\infty$ never $\frac{2}{0}$

Size → the zero Direction → the signs

$= -\infty$ → describes it, not a real limit (1.3)

IDEA 2: A VERTICAL ASYMPTOTE

2. A vertical asymptote

$x \to a$: one side → $\pm\infty$

→ $x = a$: vertical asymptote

one side is enough

Even power → one sign

$\frac{x+5}{(x-1)^2}$ → bottom > 0

Think $\frac{6}{0^+} \to +\infty$ both sides

$\lim_{x \to 1} \frac{x+5}{(x-1)^2} = \infty$

$\frac{x}{x^3} = \frac{1}{x^2}$ → asymptote

Cancel first → denominator

IDEA 3: NO FRACTION, STILL AN ASYMPTOTE

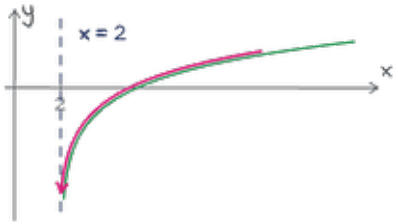
Key ideas

3. No fraction, still an asymptote

$\ln(x-2)$, $\tan x$

e.g. $\ln(x-2)$ → domain $x > 2$

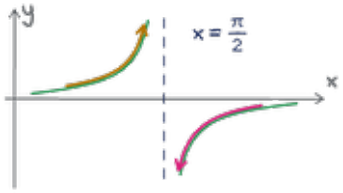
ln x, shifted right 2



$\lim_{x \to 2^+} \ln(x-2) = -\infty$

So $x = 2$ is the asymptote, not $x = 0$ (shifted)

$\tan x = \frac{\sin x}{\cos x}$ → blows up where $\cos x = 0$



Left: $\cos x \to 0^+$, up Right: $\cos x \to 0^-$, down

→ at every $x = \frac{\pi}{2} + k\pi$

$\sec x = \frac{1}{\cos x}$, $\csc x = \frac{1}{\sin x}$ → zero bottoms

PROBLEM 1 OF 2

A square in the bottom

Let $g(x) = \dfrac{3x-1}{x^2+2x+1}$.

Find $\lim_{x \to -1} g(x)$.

Explain how you know.

Scratch vs. answer

Think: $\frac{-4}{0^+} \to -\infty$

scratch only

answer → in words

Solution The top's limit, then the sign of the bottom

Plan: top limit → sign on each side

$\lim_{x \to -1} (3x-1) = 3(-1) - 1 = -4$

$x^2 + 2x + 1 = (x+1)^2$

$\lim_{x \to -1} (x+1)^2 = (-1+1)^2 = 0$

$x \neq -1 \to (x+1)^2 > 0 \to 0^+$ on both sides

Think: $\frac{-4}{0^+} \to -\infty$

Top → −4 bottom → 0 through positives

$\lim_{x \to -1} \dfrac{3x-1}{x^2+2x+1} = -\infty$

No real limit → which way → asymptote

Workbook → bottom's sign

Common mistake

X Common mistake

$\lim_{x \to -1} g(x) = \frac{-4}{0} = -\infty$

→ $\frac{-4}{0}$ is not a number

→ arithmetic with ∞

✓ Do this

Top → −4

Bottom → 0^+

$\lim_{x \to -1} g(x) = -\infty$

Think → scratch

PROBLEM 2 OF 2

Which one has the asymptote?

Which of the following functions has a vertical asymptote at $x = 3$?

(A) $\dfrac{x-3}{x^2-9}$

(B) $\dfrac{x^2-9}{x-3}$

(C) $\ln(x-3)$

(D) $\dfrac{x-3}{x^2+9}$

Solution Cancel first, then look for a side that runs off

Plan: plug in 3 → $\frac{c}{0}$, $c \neq 0$: a wall

$\frac{0}{0} \rightarrow \frac{x-3}{x-3}$

(A) $\frac{x-3}{(x-3)(x+3)} \rightarrow \frac{1}{6}$ → hole ×

(B) $\frac{(x-3)(x+3)}{x-3} \rightarrow 6$ → hole ×

(C) $x \rightarrow 3^+ \rightarrow x-3 \rightarrow 0^+$

$\lim_{x \to 3^+} \ln(x-3) = -\infty$

(D) bottom → $3^2+9 = 18 \neq 0$ ×

So (C) → $\ln(x-3)$

workbook → $\frac{x-3}{x-3(x+3)}$ → check the side

WRAP-UP

Up, or down? The signs

Up or down → signs of top and bottom, each side

Even power → one sign → same way

one side → $\pm\infty$ → vertical asymptote $x = a$

Exam → in words; never $\frac{c}{0}$

Next → 1.15 → limits at infinity

Next: the workbook for this topic (link in the video description). Stuck? Ask in the comments.