Lecture Notes · Unit 1 · Lecture 10 AP CED 1.8

Squeeze theorem

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INTRO: WHEN YOU CAN'T REWRITE IT

The Squeeze Theorem AP Calculus 1.8

Plug in → $\frac{0}{0}$ → factor, conjugate, common denominator, identity

No rewrite → limit?

Can't rewrite → trap it

$x \to 0 \to \frac{\sin x}{x} \to 1$ – and why

when it's not 1

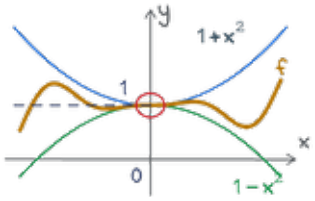
IDEA 1: TRAPPED BETWEEN TWO

Key ideas

1. Trapped between two

near 0 → $1-x^2 \le f \le 1+x^2$

Plug in 0 → as $x \to 0$, both → 1



f → ?

Stuck both → 1 no room → $f(x) \to 1$

$\lim_{x \to 0} f(x) = 1$

In general $g \le f \le h$ near c

g h → same L as $x \to c$ → $f \to L$

The squeeze theorem

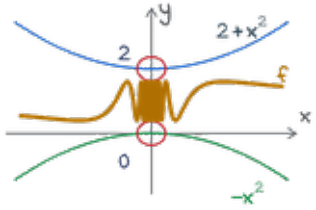
IDEA 2: CHECK BEFORE YOU SQUEEZE

Key ideas

2. Check before you squeeze

f between $-x^2$ and $2+x^2$ near 0 (including 0)

As $x \to 0$, $-x^2 \to 0$, $2+x^2 \to 2$



f could head to 1, to 1.5, or nowhere

$0 \neq 2$ → the theorem says nothing

Not 'no limit' → just no conclusion

(i) near c → $g \le f \le h$

(ii) as $x \to c$, g h → the same real L

AP → state (i), (ii), then conclude

IDEA 3: SINE STAYS BETWEEN −1 AND 1

Key ideas

3. Sine stays between −1 and 1

$-1 \le \sin \square \le 1$

e.g. $\lim_{x \to 0} x^2 \sin \frac{1}{x}$

$\sin \frac{1}{x}$ has no limit → −1 to 1

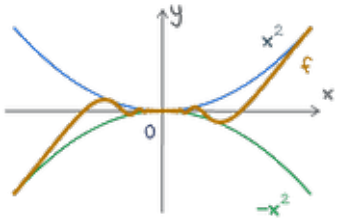
Product law needs both → no

$-1 \le \sin \square \le 1 \Rightarrow -1 \le \sin \frac{1}{x} \le 1$

$x x^2 > 0$ → order stays

$-x^2 \le x^2 \sin \frac{1}{x} \le x^2$ (i)

(ii) as $x \to 0$, $\pm x^2 \to 0$, the same real number

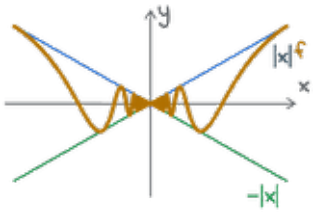


Both hold → squeeze theorem

$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$ no escape

Times x^2: $x < 0$ → order flips

$\pm x \rightarrow -|x| \le x \sin \frac{1}{x} \le |x|$ (1)



(ii) as $x \to 0$, $\pm|x| \to 0$ $\lim_{x \to 0} x \sin \frac{1}{x} = 0$

on the AP: as $x \to 0$, $\frac{\sin x}{x} \to 1$, $\frac{1-\cos x}{x} \to 0$ → why?

IDEA 4: SIN X / X AS X → 0

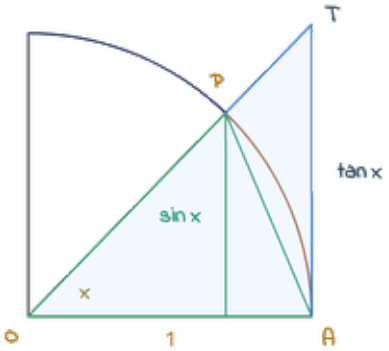
Key ideas

4. Sine x over x, as $x \to 0$

$\frac{\sin x}{x}$, $x \to 0$

Plug in 0 → top $\sin 0 = 0$, bottom 0

Can't rewrite → trap it



$0 < x < \frac{\pi}{2}$ (radians)

$\triangle OAP \rightarrow$ height $\sin x \rightarrow \frac{1}{2} \cdot 1 \cdot \sin x$

sector $OAP = \frac{1}{2} \cdot 1^2 \cdot x$

$\triangle OAT \rightarrow$ height $AT = \tan x \rightarrow \frac{1}{2} \cdot 1 \cdot \tan x$

$\triangle \le$ sector $\le \triangle$

$\frac{\sin x}{2} \le \frac{x}{2} \le \frac{\tan x}{2}$

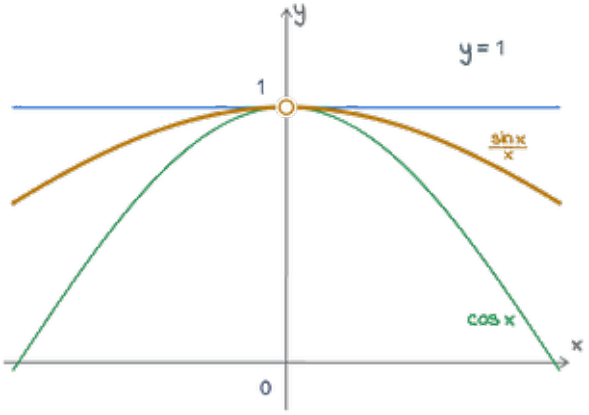
$\frac{1}{2} \sin x > 0 \rightarrow$ upper $\frac{\tan x}{\sin x}$

$1 \le \frac{x}{\sin x} \le \frac{1}{\cos x}$

All > 0: reciprocals → order flips

$\cos x \le \frac{\sin x}{x} \le 1$

As $x \to 0^+$, $\cos x \to \cos 0 = 1$



As $x \to 0^+$, bounds → 1 → squeeze: $\frac{\sin x}{x} \to 1$

$x < 0 \rightarrow x = -t$, $t > 0$ $\frac{\sin(-t)}{-t} = \frac{-\sin t}{-t} = \frac{\sin t}{t}$

$x \to 0^- \Rightarrow t \to 0^+ \rightarrow \frac{\sin t}{t} \to 1$

Both → 1 $\frac{\sin x}{x} \to 1$ as $x \to 0$

$\lim_{\square \to 0} \frac{\sin \square}{\square} = 1$ → look first

IDEAS 5–6: MATCH THE ANGLE, (1 – COS X)/X

Key ideas

5. Match the angle

e.g. $\lim_{x \to 0} \frac{\sin 3x}{x}$

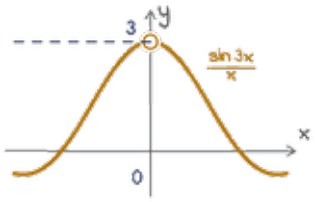
one? Not yet: angle 3x, bottom x → no match

Cross out x's → sin 3? No

$\frac{3}{3} \cdot \frac{\sin 3x}{x} = 3 \cdot \frac{\sin 3x}{3x}$

$\square = 3x;\ x \to 0 \Rightarrow 3x \to 0 \to \frac{\sin\square}{\square} \to 1$

$= 3 \cdot 1 = 3$



6. One minus cosine, over x

$\frac{1-\cos x}{x}$

$x \to 0 \to$ top $1-\cos 0 = 0$, bottom 0

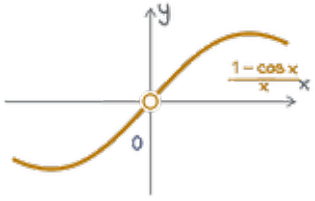
$\cos \to \sin \to 1-\cos^2 x = \sin^2 x$

$\frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x}$

$= \frac{1-\cos^2 x}{x(1+\cos x)} = \frac{\sin^2 x}{x(1+\cos x)}$

$= \left(\frac{\sin x}{x}\right)\left(\frac{\sin x}{1+\cos x}\right)$

$\to 1 \quad \to \frac{0}{1+1} = 0$



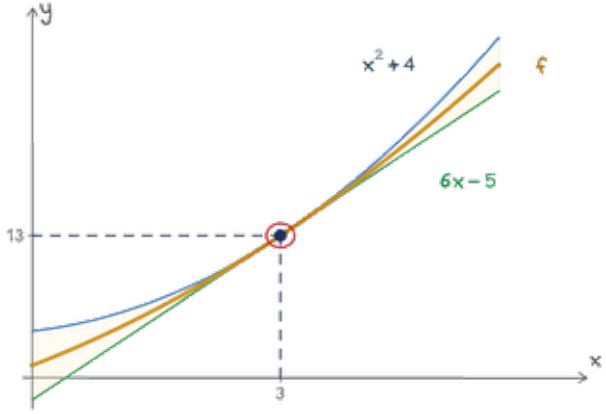
Limit = 0

Any angle $\to \frac{1-\cos\square}{\square} \to 0$ as $\square \to 0$

PROBLEM 1 OF 3

Squeeze with given bounds

For all x near 3, $6x-5 \le f(x) \le x^2+4$.
Find the limit of $f(x)$ as $x \to 3$. Justify your answer.



Solution Check both conditions, then conclude

Plan: (i), (ii), then conclude

(i) Given

Polynomials → plug in 3

$\lim_{x \to 3} (6x-5) = 13$

$3^2 + 4 = 13$

(ii) as $x \to 3$, both $\to 13$ → Squeeze theorem

$\lim_{x \to 3} f(x) = 13$

Workbook → both conditions

PROBLEM 2 OF 3

An identity, then match the angle

What is the limit of $\dfrac{1-\cos^2 3x}{2x^2}$ as $x \to 0$?

(A) 0

(B) $\frac{3}{2}$

(C) $\frac{9}{2}$

(D) nonexistent

Identity

$1-\cos^2\square = \sin^2\square$

Special limit

$\lim_{\square\to 0}\dfrac{\sin\square}{\square} = 1$

Solution Identity first, then match the angle

Plug in → identity → match angle

$1-\cos^2 0 = 0,\ 2(0)^2 = 0$

$$\lim_{x\to 0}\frac{\sin^2 3x}{2x^2}$$

$$= \lim_{x\to 0}\frac{1}{2}\left(\frac{\sin 3x}{x}\right)^2 = \lim_{x\to 0}\frac{1}{2}\left(3\cdot\frac{\sin 3x}{3x}\right)^2$$

$$= \lim_{x\to 0}\frac{1}{2}\cdot 3^2\left(\frac{\sin 3x}{3x}\right)^2 = \lim_{x\to 0}\frac{9}{2}\left(\frac{\sin 3x}{3x}\right)^2$$

as $x \to 0$, $3x \to 0$ → fraction → 1; so $\frac{9}{2}\cdot 1^2 = \frac{9}{2}$

So (C) → $\frac{9}{2}$

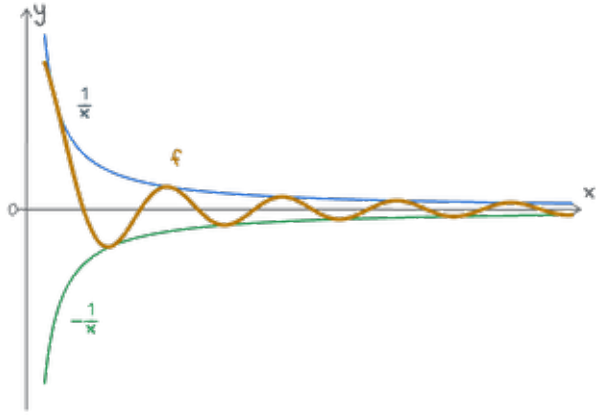
Workbook → identity first

PROBLEM 3 OF 3

sin x / x as x goes to infinity

Find $\lim_{x\to\infty}\dfrac{\sin x}{x}$.

Justify your answer.



Solution Check where the box goes, then squeeze

Plan: $\square = x \to \infty$, not 0 → no 1 rule → squeeze

$-1 \le \sin x \le 1$

(i) $x > 0 \to -\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$

(ii) as $x \to \infty$, $\pm\frac{1}{x} \to 0$, the same real number

Squeeze theorem →

Limit = 0 ≠ 1

Workbook → (i), (ii) again

Common mistake

X Common mistake

$\lim_{x\to \pi/12}\dfrac{\sin 6x}{x} = 6$

→ treated as $x \to 0$

→ only if $\square \to 0$

✓ Do this

$\square = 6x \to \frac{\pi}{2}$, not 0 → plug in

$$= \frac{\sin\frac{\pi}{2}}{\frac{\pi}{12}} = \frac{1}{\frac{\pi}{12}} = \frac{12}{\pi}$$

WRAP-UP



Can't rewrite it? Trap it

Trap → check (i), (ii) → conclude

$\cos x \le \frac{\sin x}{x} \le 1$ → the first limit

$\frac{1-\cos\square}{\square} \to 0$

Use them only as $\square \to 0$ → look first

Next → 1.10 → types of discontinuities

Next: the workbook for this topic (link in the video description). Stuck? Ask in the comments.