1.8 / The Squeeze Theorem

Determining limits using the squeeze theorem

Watch the lesson ➤

Learning goal
Find a limit by trapping a function between two others, and use the two special limits of sine and cosine.

Key ideas

01 Trapped between two: e.g. $1 - x^2 \le f(x) \le 1 + x^2$ near 0. As $x \to 0$, both polynomials head to 1 (plug in 0). No room in the middle, so $\lim_{x\to 0} f(x) = 1$. In general: if $g \le f \le h$ near c, and as $x \to c$, $g \to L$ and $h \to L$, then $f \to L$. The **squeeze** theorem.

02 Check before you squeeze: with $-x^2 \le f(x) \le 2 + x^2$ near 0, as $x \to 0$, $-x^2 \to 0$ but $2 + x^2 \to 2$. A gap from 0 to 2: f could head to 1, to 1.5, or nowhere, so the theorem says **nothing**. Check (i) $g \le f \le h$ near c; (ii) as $x \to c$, $\lim g = \lim h = L$, a real number. On the AP exam, state (i) and (ii), then conclude.

03 Sine stays between -1 and 1: for $\lim_{x\to 0} x^2 \sin\frac{1}{x}$, $\sin\frac{1}{x}$ alone has no limit (it swings), but $-1 \le \sin\frac{1}{x} \le 1$. Times $x^2 > 0$, the order stays: (i) $-x^2 \le x^2 \sin\frac{1}{x} \le x^2$. (ii) As $x \to 0$, $-x^2 \to 0$ and $x^2 \to 0$, the same real number. By the squeeze theorem, the limit is 0. Between fixed numbers is **bounded**: bounded times something heading to 0 heads to 0. Times x, which may be negative: $-|x| \le x \sin\square \le |x|$.

04 $\frac{\sin x}{x}$ as $x \to 0$: plugging in gives $\frac{0}{0}$ and no rewrite works, so trap it. On the unit circle, angle x, $0 < x < \frac{\pi}{2}$ (radians): triangle $OAP \le$ sector $OAP \le$ triangle OAT, so $\frac{1}{2}\sin x \le \frac{1}{2}x \le \frac{1}{2}\tan x$. Divide by $\frac{1}{2}\sin x > 0$: $1 \le \frac{x}{\sin x} \le \frac{1}{\cos x}$. All positive, so flip all three: $\cos x \le \frac{\sin x}{x} \le 1$. As $x \to 0^+$, $\cos x \to 1$; for $x < 0$ it is the same. So $\lim_{x\to 0} \frac{\sin x}{x} = 1$, for any angle: $\lim_{\square\to 0} \frac{\sin\square}{\square} = 1$. Does $\square \to 0$? Yes: 1. No: plug in.

05 Match the **angle**: $\lim_{x\to 0} \frac{\sin 3x}{x} = \lim_{x\to 0} 3 \cdot \frac{\sin 3x}{3x} = 3 \cdot 1 = 3$, since $\square = 3x \to 0$.

06 $\lim_{x\to 0} \frac{1-\cos x}{x}$: multiply by $\frac{1+\cos x}{1+\cos x}$ to get $\frac{\sin^2 x}{x(1+\cos x)} = \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x}$, so the limit is $1 \cdot 0 = \mathbf{0}$. Any angle: $\lim_{\square\to 0} \frac{1-\cos\square}{\square} = 0$.

Remember

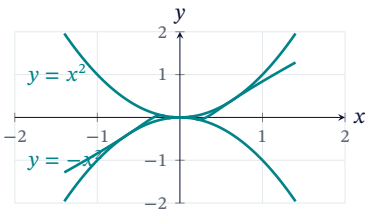
squeeze theorem
If $g \le f \le h$ near c and g, h have the same limit L at c, then f has limit L.

bounding functions
The two functions g and h with $g(x) \le f(x) \le h(x)$.

bounded
Stays between two fixed numbers, as $-1 \le \sin x \le 1$.

radian
The angle unit of calculus. $\frac{\sin x}{x} \to 1$ needs x in radians.

Squeezed to the same limit



For $x \neq 0$, $-x^2 \le x^2 \sin\frac{1}{x} \le x^2$. As $x \to 0$, both bounds approach 0, so the middle graph does too.

Class notes

1.8 / Write it clearly

What earns the point on the AP Exam, and the mistakes that lose it.

Writing checklist

Notation to use and to avoid

WRITE	AVOID
State both conditions before the conclusion: "$g(x) \le f(x) \le h(x)$ near c, and $\lim g(x) = \lim h(x) = L$, so by the squeeze theorem $\lim f(x) = L$."	Do not write a limit equal to $\frac{0}{0}$. The 0/0 form is not a value.
Show the limits of the two bounds with lim: $\lim_{x\to 3}(6x-5) = 13$, not only the values at 3.	Do not drop lim before the last step: $\lim_{x\to 0} \frac{5}{2} \cdot \frac{\sin 5x}{5x} = \frac{5}{2}$.

Common mistakes

The mistake code is used in the answers and in My Mistakes

01 Match the angle

Code L7

✗ $\lim_{x\to 0} \dfrac{\sin 10x}{x} = 1$, because sin(box)/box goes to 1.

✓ $\dfrac{\sin 10x}{x} = 10 \cdot \dfrac{\sin 10x}{10x}$, and $\dfrac{\sin 10x}{10x} \to 1$ as $x \to 0$: the limit is 10.

02 Only as the angle approaches 0

Code L7

✗ $\lim_{x\to \pi/2} \dfrac{\sin x}{x} = 1$.

✓ The angle approaches $\frac{\pi}{2}$, not 0, and the bottom does not approach 0: substitute. The limit is $\dfrac{\sin(\pi/2)}{\pi/2} = \dfrac{2}{\pi}$.

03 Use |x| when the factor can be negative

Code L4

✗ Since $-1 \le \cos\frac{2}{x} \le 1$, we have $-x \le x\cos\frac{2}{x} \le x$ for all $x \neq 0$.

✓ For $x < 0$ the inequalities flip. Use $-|x| \le x\cos\frac{2}{x} \le |x|$; both bounds approach 0, so the limit is 0.

Real AP questions on this topic Free on AP Central. Try them after the practice.

No released FRQ from 2023–2026 tests the squeeze theorem. The CED sample multiple-choice question 1 uses a Pythagorean identity and then a sine limit with a matched angle.

1.8 / Guided examples

Watch, pause, and write the reasoning. Full solutions are in the video.

Problem types

Read the problem, name the type, then use its method

Type	Method	Problems
Squeeze with given bounds	Check (i) $g \le f \le h$ near c and (ii) $\lim g = \lim h = L$, the same real number. Then $\lim f = L$.	E01 · Practice 1
A bounded factor times a small factor	Start from $-1 \le \sin(\square) \le 1$; multiply by x^2 or $\lvert x\rvert$, which is never negative; squeeze.	Practice 2
Match the angle in sin(box)/box	Make the bottom equal to the angle: $\frac{\sin(\square)}{\square} \to 1$ when $\square \to 0$.	Practice 3
The cosine limit (1 − cos x)/x	$\frac{1-\cos(\square)}{\square} \to 0$ when $\square \to 0$. Constants come out by the limit laws.	Practice 4
Rewrite, then use the special limits	Use an identity ($\tan = \frac{\sin}{\cos}$, $\sin^2 + \cos^2 = 1$), split into factors with known limits, use the product law.	E02
Look where x goes (and from which side)	See where the angle goes. To 0: match it. To a number that is not 0, with a bottom that does not approach 0: substitute. Unbounded, as $\frac{6}{x}$ when $x \to 0$: a bounded factor times a factor that approaches 0; squeeze, with $\lvert x\rvert$ when the factor can be negative.	E03 · Practice 6
Do the conditions hold? (graphs and words)	Read $\lim g$ and $\lim h$. Equal real numbers: squeeze. Different: the theorem gives no answer.	Practice 5

Example 01

Video · E01

For all x near 3, $6x - 5 \le f(x) \le x^2 + 4$.
Find the limit of $f(x)$ as $x \to 3$. Justify your answer.

> Do the two outside functions head to the same number?

Example 02

Video · E02

What is the limit of $\dfrac{1-\cos^2 3x}{2x^2}$ as $x \to 0$?

A. 0
B. $\dfrac{3}{2}$
C. $\dfrac{9}{2}$
D. nonexistent

> Substitute first. With 0/0 and trig, use an identity on the top.

Example 03

Video · E03

Find $\lim_{x\to\infty} \dfrac{\sin x}{x}$.
Justify your answer.

> Where does the angle go? Not to 0.

VIDEO COMPLETE Check your example answers at the end of this booklet.

1.8 / Example answers

The Squeeze Theorem

E01 13

E02 (C)

E03 0