Lecture Notes · Unit 1 · Lecture 8 AP CED 1.6b

Conjugates & trig identities

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INTRO: WHEN 0/0 WON'T FACTOR

Conjugates & Trig Identities AP Calculus 1.6

$\frac{0}{0}$ → last time: factor, cancel

No polynomial?

$\frac{0}{0}$ → not done yet (not an answer, not "DNE")

AP → 3 rewrites: factor, conjugate, trig

Factor ✓ → today: 2 more

hidden factor?

THE CONJUGATE, THEN A ROOT ON TOP

A root on top

A tool: make a root disappear

$(\sqrt{A})^2 = A$ → root gone

x ?

$(a-b)(a+b) = a^2 - b^2$ (difference of squares)

$\sqrt{A} - b \to \sqrt{A} + b$

$(\sqrt{A})^2 - b^2 = A - b^2$

Partner = the conjugate

$\frac{\sqrt{x+3}-2}{x-1}$ as $x \to 1$

$\sqrt{1+3} - 2 = \sqrt{4} - 2 = 0, \quad 1 - 1 = 0$

want $(x-1)$ on top → hides under the root

$a = \sqrt{x+3}, b = 2$

$(\sqrt{x+3}-2)(\sqrt{x+3}+2)$

$= (\sqrt{x+3})^2 - 2^2$

$= (x+3) - 4 = x - 1$

A ROOT: TOP AND BOTTOM

A root: top and bottom

x conjugate

$x\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}$ $(x \to 1)$

Keep the bottom factored → don't multiply it out

$$\lim_{x\to 1} \frac{(x-1)}{(x-1)(\sqrt{x+3}+2)}$$

$$= \lim_{x\to 1} \frac{1}{\sqrt{x+3}+2}$$

Bottom → $\sqrt{4} + 2 = 4 \neq 0$ → plug in

$= \frac{1}{2+2} = \frac{1}{4}$

As $x \to a$, $\frac{0}{0}$ from $\sqrt{\text{polynomial}} - \text{number}$

→ x conjugate → polynomial

Zero at a → $(x-a)$ shows up

TRIG: ANOTHER FORM

Trig another form

$$\lim_{x\to 0} \frac{1-\cos^2 x}{\sin x}$$

$1 - \cos^2 0 = 1 - 1 = 0, \quad \sin 0 = 0$

Bottom $\sin x \to 0$ → look for $\sin x$ on top

$1 - \cos^2 x$ → all cosine? $\sin^2 x + \cos^2 x = 1$

→ $1 - \cos^2 x = \sin^2 x$

$$\lim_{x\to 0} \frac{\sin^2 x}{\sin x}$$

$x \neq 0$ zeros at $n\pi$ → near 0, $\sin x \neq 0$

$= \lim_{x \to 0} \sin x$

Defined at 0 → plug in

$\lim_{x \to 0} \sin x = \sin 0 = 0$

0 = fine → $\frac{0}{0}$ = unfinished

Every x → identity

→ $\tan x = \frac{\sin x}{\cos x}$ ($\cos x \neq 0$) → another identity

PROBLEM 1 OF 3

A root on the bottom

Find the limit of $\dfrac{x-16}{\sqrt{x}-4}$ as $x \to 16$.

Difference of squares

$(a-b)(a+b) = a^2 - b^2$

Solution

Multiply the top and the bottom by the conjugate

Plan: root below → conjugate, top & bottom

$16 - 16 = 0, \sqrt{16} - 4 = 0$

$\times \frac{\sqrt{x}+4}{\sqrt{x}+4}$ (= 1)

$(\sqrt{x}-4)(\sqrt{x}+4) = x - 16$

$x - 16 \neq 0$

$$\lim_{x \to 16} \frac{(x-16)(\sqrt{x}+4)}{(x-16)}$$

$$= \lim_{x \to 16} (\sqrt{x}+4) = 4 + 4 = 8$$

Or: $\frac{(\sqrt{x}-4)(\sqrt{x}+4)}{(\sqrt{x}-4)}$ → same 8

Workbook → its conjugate

Careful → where points are lost

✗ Common mistake

Bottom only $x(\sqrt{x}+4)$

$\lim_{x \to 16} \frac{x-16}{x-16} = 1$

$1 \neq 8$ → a different function

✓ Do this

Top & bottom → ×1

→ limit stays 8

PROBLEM 2 OF 3

A trig difference of squares

Find the limit of $\dfrac{\sin x - \cos x}{\sin^2 x - \cos^2 x}$ as $x \to \dfrac{\pi}{4}$.

Difference of squares

$a^2 - b^2 = (a-b)(a+b)$

Solution Factor a difference of squares

Plan: plug in → $\frac{0}{0}$? factor

sin, cos at $\frac{\pi}{4}$ → $\frac{\sqrt{2}}{2}$

Top → $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = 0$

$\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} = \frac{1}{2}$

Bottom → $\frac{1}{2} - \frac{1}{2} = 0$

$\sin^2 x - \cos^2 x = (\sin x - \cos x)(\sin x + \cos x)$

Near $\frac{\pi}{4}$, $x \neq \frac{\pi}{4}$ → $\sin x - \cos x \neq 0$

Solution Factor a difference of squares

Difference of squares

$a^2 - b^2 = (a-b)(a+b)$

Plan: cancel → plug in

$$\lim_{x \to \frac{\pi}{4}} \frac{A}{A(\sin x + \cos x)}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{1}{\sin x + \cos x}$$

Bottom → $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \neq 0$ → plug in

$$= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Workbook → find the factor

PROBLEM 3 OF 3

Tangent, then an identity

What is the limit of $\dfrac{1-\cos x}{\sin x \tan x}$ as $x \to 0$?

(A) 2

(B) 1

(C) $\frac{1}{2}$

(D) nonexistent

Identity

$\sin^2 x + \cos^2 x = 1$

Solution Tangent as sine over cosine, then the identity

Plan: tan = $\frac{\sin}{\cos}$ → identity

$1 - \cos 0 = 0$, $\sin 0 \cdot \tan 0 = 0$

$\frac{0}{0}$ → D? No → not done yet, rewrite

Bottom = $\sin x \cdot \frac{\sin x}{\cos x}$

$$\lim_{x \to 0} \frac{1-\cos x}{\sin x \tan x} \cdot \frac{\cos x}{\cos x} = \lim_{x \to 0} \frac{(1-\cos x)\cos x}{\sin^2 x}$$

$(1 - \cos x)$ makes the 0 → bottom in cosines too

$\sin^2 x = 1 - \cos^2 x = (1-\cos x)(1+\cos x)$

Solution Tangent as sine over cosine, then the identity

Identity

$\sin^2 x + \cos^2 x = 1$

Plan: cancel → plug in

$\cos x = 1$ only at $2n\pi$

$x \to 0, x \neq 0 \to 1 - \cos x \neq 0$

$$\lim_{x \to 0} \frac{(1-\cos x)\cos x}{(1-\cos x)(1+\cos x)}$$

$$= \lim_{x \to 0} \frac{\cos x}{1+\cos x}$$

Bottom → $1 + 1 = 2 \neq 0$ → plug in

$$= \frac{1}{1+1} = \frac{1}{2}$$

So (C)

Workbook → rewrite first

WRAP-UP

How do we bring out the factor?

$\frac{0}{0}$ → not done yet; AP's 3 rewrites

A root → its conjugate

Top and bottom; keep it factored

Trig → $\sin^2 x + \cos^2 x = 1$, $\tan x = \frac{\sin x}{\cos x}$

Rewrite until it shows → cancel → plug in

Next: 1.7 → which method? fraction in a fraction