1.6b / Conjugates and Trig Identities

Determining limits using algebraic manipulation · Part 2

Watch the lesson ➤

Learning goal

Find limits where direct substitution gives the 0/0 form by using a conjugate or a trigonometric identity.

Key ideas

01 A root on top: $\lim_{x\to 1}\dfrac{\sqrt{x+3}-2}{x-1}$. The top heads to 0 at 1, but $(x-1)$ hides under the root, so factoring will not work. Square a root and it is gone: $(\sqrt{A})^2 = A$. With $a = \sqrt{x+3}$ and $b = 2$, $\sqrt{x+3}+2$ is the **conjugate**: the same two terms, the middle sign flipped.

02 Multiply the top and the bottom by the conjugate (the same top and bottom, so times 1): $\dfrac{(x+3)-4}{(x-1)(\sqrt{x+3}+2)} = \dfrac{x-1}{(x-1)(\sqrt{x+3}+2)}$. $x \to 1$ means $x \neq 1$, so $x-1 \neq 0$ cancels. The bottom heads to $\sqrt{4}+2 = 4 \neq 0$, so plug in: the limit is $\dfrac{1}{4}$. In general, for $\dfrac{0}{0}$ with a root, use the conjugate on the top and the bottom until $(x-a)$ shows.

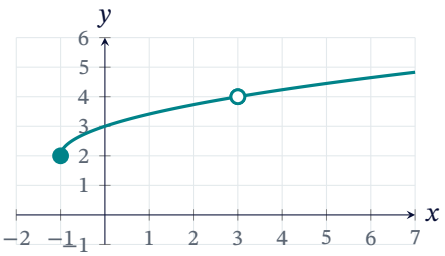
03 Trig on top: $\lim_{x\to 0}\dfrac{1-\cos^2 x}{\sin x}$. The bottom $\sin x \to 0$, so look for $\sin x$ on top. $\sin^2 x + \cos^2 x = 1$ (the unit circle), so $1-\cos^2 x = \sin^2 x$, and $\dfrac{\sin^2 x}{\sin x} = \sin x$, since near 0, $x \neq 0$, $\sin x \neq 0$. Sine has no breaks: plug in, the limit is $\sin 0 = 0$. True for every x: an **identity**.

Remember

conjugate
For $\sqrt{A}-b$, the conjugate is $\sqrt{A}+b$. Their product has no root.

identity
An equation true for every x where both sides are defined, such as $\sin^2 x + \cos^2 x = 1$.

The conjugate keeps the value



For $x \neq 3$, $\dfrac{x-3}{\sqrt{x+1}-2} = \sqrt{x+1}+2$: the same curve with an open circle at $(3,4)$. The limit is 4.

Class notes

1.6b / Write it clearly

What earns the point on the AP Exam, and the mistakes that lose it.

Writing checklist

Notation to use and to avoid

WRITE	AVOID
Say what substitution gives: "Direct substitution gives the 0/0 form, so we multiply by the conjugate."	Do not write a limit equal to $\frac{0}{0}$. The 0/0 form is not a value.
Keep lim on every line: $\lim_{x\to 8} \dfrac{1}{\sqrt{x+8}+4} = \dfrac{1}{8}$.	

Common mistakes

The mistake code is used in the answers and in My Mistakes

01 Multiply the top and the bottom by the conjugate

Code L6

✗ Multiplying only below in $\dfrac{x-16}{\sqrt{x}-4}$ gives $\dfrac{x-16}{x-16} = 1$.

✓ Multiply by $\dfrac{\sqrt{x}+4}{\sqrt{x}+4}$ to preserve the expression. The simplified form $\sqrt{x}+4$ approaches 8 at 16.

02 Keep the other factor

Code L6

✗ $\dfrac{\sqrt{x+20}-5}{x-5} = \dfrac{x-5}{(x-5)\cdot 5}$, so the limit at 5 is $\frac{1}{5}$.

✓ The other factor is $\sqrt{x+20}+5$, which approaches 10: the limit is $\frac{1}{10}$.

03 Square both terms

Code L6

✗ $(\sqrt{x+5}-3)(\sqrt{x+5}+3) = x+5-3$.

✓ $(\sqrt{x+5}-3)(\sqrt{x+5}+3) = x+5-9 = x-4$.

04 Use an identity, not a crossed-out word

Code L5

✗ $\dfrac{\sin 4x}{\sin 2x} = \dfrac{4x}{2x} = 2$ (crossing out "sin").

✓ $\sin 4x = 2\sin 2x \cos 2x$, so $\dfrac{\sin 4x}{\sin 2x} = 2\cos 2x$ when $\sin 2x \neq 0$.

Real AP questions on this topic Free on AP Central. Try them after the practice.

No released FRQ asks only for a conjugate or identity limit. The CED names conjugates and alternate forms of trigonometric functions as examples for Topic 1.6.

1.6b / Guided examples

Watch, pause, and write the reasoning. Full solutions are in the video.

Problem types

Read the problem, name the type, then use its method

Type	Method	Problems
Multiply by the conjugate (a root in the top)	The 0/0 form with a root: multiply top and bottom by the conjugate, cancel, substitute.	
A root in the bottom	Multiply top and bottom by the conjugate of the bottom.	E01 · Practice 1
A root of a − x (watch the sign)	After the conjugate, factor the new top; keep the negative sign and any coefficient (e.g. $-x$, $-2x$, $-(x-a)$).	
Trigonometric identities	Rewrite with $\sin^2 x + \cos^2 x = 1$, $\sin 2x = 2\sin x \cos x$, or $\tan x = \frac{\sin x}{\cos x}$, cancel, substitute.	E02, E03 · Practice 2
Rewrite an average-rate expression with a conjugate	Keep the time interval h nonzero while simplifying the average rate of change. Then examine the simplified expression as the gap approaches zero.	Practice 4
The limit and the value at the same point	The limit comes from the rewritten form; the value at a comes from the original definition (undefined there unless it is given separately). Cancelling never creates a value.	Practice 3

Example 01

Video · E01

Find the limit of $\dfrac{x-16}{\sqrt{x}-4}$ as $x \to 16$.

> Plug in first. Then the bottom's conjugate, on the top and the bottom.

Example 02

Video · E02

Find the limit of $\dfrac{\sin x - \cos x}{\sin^2 x - \cos^2 x}$ as $x \to \dfrac{\pi}{4}$.

> The bottom is a difference of squares.

Example 03

Video · E03

What is the limit of $\dfrac{1 - \cos x}{\sin x \tan x}$ as $x \to 0$?

A. 2
B. 1
C. $\dfrac{1}{2}$
D. nonexistent

> Write tan x as sin x over cos x. Then use $\sin^2 x = 1 - \cos^2 x$.

VIDEO COMPLETE Check your example answers at the end of this booklet.

1.6b / Example answers

Conjugates and Trig Identities

E01 8

E02 $\frac{\sqrt{2}}{2}$

E03 (C)