Lecture Notes · Unit 1 · Lecture 7 AP CED 1.6a

Limits by factoring

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INTRO: IS THE LIMIT STUCK?

Limits by Factoring AP Calculus 1.6

top = numerator, bottom = denominator

Plug in → $\frac{0}{0}$ → stuck?

$\frac{0}{0}$ → a signal → factor

Every line true: never "$= \frac{0}{0}$"

First, why: real limit, bottom → 0 → top → 0

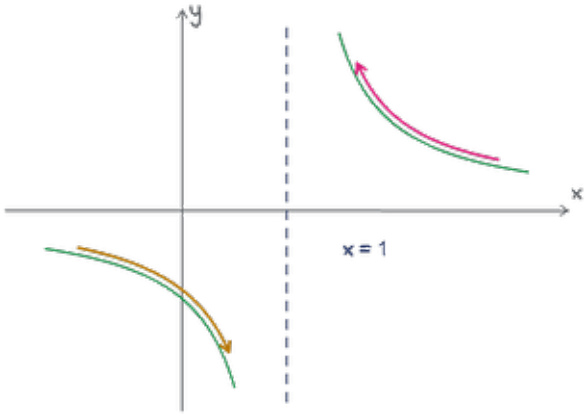
IDEA 1: A NUMBER OVER ZERO

Key ideas

1. If the bottom → 0, what does the top do?

e.g. $\frac{3}{x-1}$ as $x \to 1$

Top → 3, bottom → 0



Graph: dashed line $x = 1$

$x \to 1^+, x > 1 \to x - 1 \to 0^+$

$x \to 1^-, x < 1 \to x - 1 \to 0^-$

Think: $\frac{3}{0^+} \to \infty$, $\frac{3}{0^-} \to -\infty$

Opposite ways → no limit

Any $\frac{c}{x-a}$, $c \neq 0$, as $x \to a$ → no real limit

IDEA 2: WHY THE TOP MUST HEAD TO 0

Key ideas

2. The limit of a fraction

$\lim_{x \to a} \frac{\text{top}}{\text{bottom}} = L$

Bottom → 0 → top → □

$\text{top} = \frac{\text{top}}{\text{bottom}} \cdot \text{bottom}$

$\frac{\text{top}}{\text{bottom}} \to L$ bottom → 0

top → $L \cdot 0 = 0$

$\frac{0}{0}$ → indeterminate form

L finite, bottom → 0 ⇒ top → 0

FACTOR, CANCEL, PLUG IN

Key ideas

e.g. $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$

$\lim_{x \to 1} (x^2 - 1) = 1^2 - 1 = 0$

$\lim_{x \to 1} (x - 1) = 1 - 1 = 0$

→ $(x - 1)$ is a factor (factor theorem)

$x^2 - 1 = (x - 1)(x + 1)$

$x \to 1$ means $x \neq 1 \to x - 1 \neq 0$ → cancel

$\lim_{x \to 1} \frac{x-1}{x-1}(x + 1)$

$= \lim_{x \to 1} (x + 1) = 2$

IDEA 3: X → A, FIND THE FACTOR (X − A)

Key ideas

3. $x \to 4$ → find the factor $(x-4)$

$(x-4)$

e.g. $\lim_{x\to 4} \frac{x^2-16}{4x-x^2}$

$4^2 - 16 = 0,\ 4(4) - 4^2 = 0$

Both 0 at 4 → $(x-4)$ in both

Cancel a whole factor, not a term

$x^2 - 16 = (x-4)(x+4)$

$4x - x^2$ → no $(x-4)$? $4x - x^2 = x(4-x)$

$4 - x = -(x-4) \Rightarrow 4x - x^2 = -x(x-4)$

$$\to \lim_{x\to 4} \frac{(x-4)(x+4)}{-x(x-4)}$$

$$= \lim_{x\to 4} \frac{x+4}{-x} = \frac{8}{-4} = -2$$

Polynomials $\frac{0}{0}$ at a → look for $(x-a)$ as $x \to a$

IDEA 4: THE HOLE STAYS

Key ideas

4. After canceling the value at the point?

$\frac{x^3-27}{x-3}$ as $x \to 3$

$3^3 - 27 = 0,\ 3 - 3 = 0$

So $x^3 - 27 = (x-3)(\ ?\ )$

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$27 = 3^3 \Rightarrow (x-3)(x^2+3x+9)$

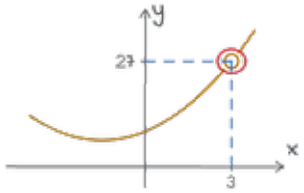
$$\to \lim_{x\to 3} \frac{(x-3)(x^2+3x+9)}{x-3}$$

$$= \lim_{x\to 3} (x^2+3x+9) = 9+9+9 = 27$$

At 3 → bottom $3 - 3 = 0$

→ undefined at 3

$$y = \frac{x^3-27}{x-3} = x^2+3x+9 \ (x \neq 3)$$



Cancel → limit; hole stays → two questions

PROBLEM 1 OF 2

Make the limit exist

The limit $\lim_{x\to 2} \frac{x^2+ax-6}{x-2}$ exists.

Find a and the value of the limit.

Finite limit, bottom → 0
⇒ top → 0

Solution

The bottom heads to 0 and the limit exists, so the top heads to 0

Plan: bottom → 0, limit exists → top → 0 → a

$$\lim_{x\to 2} (x-2) = 0$$

$$\lim_{x\to 2} (x^2+ax-6) = 4+2a-6 = 0$$

$$2a - 2 = 0 \ \to \ a = 1$$

Top → 0 → necessary, not enough → factor and check

$$x^2+x-6 = (x-2)(x+3)$$

$$x \neq 2 \to x-2 \neq 0$$

$$\lim_{x\to 2} \frac{x^2+x-6}{x-2} = \lim_{x\to 2} \frac{x-2}{x-2}(x+3)$$

$$= \lim_{x\to 2} (x+3) = 5$$

$a = 1$, limit 5 (real)

Workbook → top to 0 first

Careful → never "$= \frac{0}{0}$"

✗ Common mistake

$\lim_{x \to 2} \frac{x^2+x-6}{x-2} = \frac{0}{0}$

→ stop

$\frac{0}{0}$ not a number → false

✓ Do this

Top's limit = 0

Bottom's limit = 0

Then factor

PROBLEM 2 OF 2

Cubes and squares

Let $g(x) = \dfrac{x^3 - 8}{x^2 - 4}$.

Find the limit of $g(x)$ as $x \to 2$. Show the work that leads to your answer.

$$a^3 - b^3 = (a-b)(a^2+ab+b^2)$$
$$a^2 - b^2 = (a-b)(a+b)$$

Solution Factor the top and the bottom, cancel, plug in

Top, bottom limits → factor → cancel, plug in

$\lim_{x \to 2} (x^3 - 8) = 0 \quad \lim_{x \to 2} (x^2 - 4) = 0$

$(x-2)$ in each

$x^3 - 8 = (x-2)(x^2+2x+4)$

$x^2 - 4 = (x-2)(x+2)$

$x - 2 \neq 0$

$$\lim_{x \to 2} \frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)}$$

$$= \lim_{x \to 2} \frac{x^2+2x+4}{x+2}$$

Bottom → $4 \neq 0$ → plug in

$$= \frac{4+4+4}{2+2} = \frac{12}{4} = 3$$

Workbook → every line

WRAP-UP

Is the limit stuck?

No → $\frac{0}{0}$ = a locked door, not a wall

Key → as $x \to a$, find $(x-a)$, cancel, plug in again

The hole stays → the value may be undefined

Exam → top's limit, bottom's limit; never "$= \frac{0}{0}$"

Workbook · comments → Next: 1.6, nothing to factor?

Next: the workbook for this topic (link in the video description). Stuck? Ask in the comments.