1.6a / Limits by Factoring

Watch the lesson ➤

Determining limits using algebraic manipulation · Part 1

Learning goal

Find limits where direct substitution gives the 0/0 form by factoring and cancelling, and keep the limit and the value apart.

Key ideas

01 Top $\to c \neq 0$, bottom $\to 0$: e.g. $\lim_{x\to 1} \dfrac{3}{x-1}$. The top heads to 3, the bottom to 0. Think: $\dfrac{3}{0^+} = \infty$, $\dfrac{3}{0^-} = -\infty$. No single real number, so **no** limit.

02 So if a limit is a real number L and the bottom $\to 0$, the **top** $\to 0$ too: for $x \neq 1$, $\text{top} = \dfrac{\text{top}}{\text{bottom}} \cdot \text{bottom}$, so $\lim \text{top} = L \cdot 0 = 0$. A polynomial with a zero at 1 has the factor $(x-1)$ (factor theorem). E.g. $\lim_{x\to 1} \dfrac{x^2-1}{x-1} = \lim_{x\to 1}(x+1) = 2$, since $x \to 1$ means $x \neq 1$, so $x - 1 \neq 0$ cancels.

03 Find $(x-a)$, even written backwards: $\lim_{x\to 4} \dfrac{x^2-16}{4x-x^2}$. Both are 0 at 4, so both have $(x-4)$: $x^2 - 16 = (x-4)(x+4)$, $4x - x^2 = -x(x-4)$, and the limit is $\dfrac{8}{-4} = -2$. Cancel a whole **factor**, never a term. In general, for polynomials with 0/0 at a, look for $(x-a)$ in any form.

04 A cube on top: $\lim_{x\to 3} \dfrac{x^3-27}{x-3}$. With $a^3 - b^3 = (a-b)(a^2+ab+b^2)$, $x^3 - 27 = (x-3)(x^2+3x+9)$, so the limit is $9 + 9 + 9 = 27$.

05 The hole stays: $f(x) = \dfrac{x^2-1}{x-1}$ has $\lim_{x\to 1} f(x) = 2$, but in f the bottom at 1 is $1 - 1 = 0$, so $f(1)$ is undefined. The graph is $y = x + 1$ with a hole at $(1, 2)$. Cancelling gives the limit; the **hole** stays.

Remember

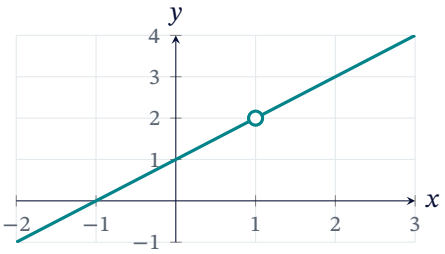
0/0 form
What direct substitution gives when the top and the bottom both approach 0. It is not a value; it means rewrite.

factor
An expression multiplied by another one, such as (x − 4) in (x − 4)(x + 3).

cancel
Divide the top and the bottom by the same nonzero factor.

hole
A missing point of the graph; the limit can exist there even though the value does not.

Cancelling leaves a hole



The graph of $f(x) = \dfrac{x^2-1}{x-1}$ is the line $y = x + 1$ with an open circle at $(1, 2)$: the limit is 2, and $f(1)$ is undefined.

Class notes

1.6a / Write it clearly

What earns the point on the AP Exam, and the mistakes that lose it.

Writing checklist

Notation to use and to avoid

WRITE	AVOID
Say what substitution gives, in words: "Direct substitution gives the form 0/0, so we factor."	Do not write a limit equal to $\frac{0}{0}$. The 0/0 form is not a value.
Keep lim until you substitute: $\lim_{x\to 4} \frac{(x-4)(x+3)}{x-4} = \lim_{x\to 4}(x+3) = 7$.	Do not drop lim after factoring: writing $\frac{(x-4)(x+3)}{x-4} = x+3 = 7$ is wrong notation.

Common mistakes

The mistake code is used in the answers and in My Mistakes

01 The 0/0 form is not the answer

Code L1

✗ Substitution in $\frac{x^2+x-20}{x-4}$ gives the 0/0 form, so the limit does not exist.

✓ Factor: $\lim_{x\to 4} \frac{(x-4)(x+5)}{x-4} = \lim_{x\to 4}(x+5) = 9$.

02 Watch the sign of a backwards factor

Code L4

✗ $\frac{x-7}{49-x^2} = \frac{1}{x+7}$, so the limit at 7 is $\frac{1}{14}$.

✓ $49-x^2 = -(x-7)(x+7)$, so the expression is $-\frac{1}{x+7}$ and the limit is $-\frac{1}{14}$.

03 Cancel factors, not terms

Code L5

✗ In $\frac{x^2+3x}{x^2-9}$, cross out x^2 to get $\frac{3x}{-9}$.

✓ Factor first: $\frac{x(x+3)}{(x-3)(x+3)} = \frac{x}{x-3}$ for $x \neq -3$.

Real AP questions on this topic Free on AP Central. Try them after the practice.

No released FRQ asks only for a factor-and-cancel limit. In the 2023 AB4(c) scoring guidelines, a response that set a limit equal to 0/0 did not earn the first point (a Unit 4 question).

1.6a / Guided examples

Watch, pause, and write the reasoning. Full solutions are in the video.

Problem types

Read the problem, name the type, then use its method

Type	Method	Problems
Factor and cancel (a quadratic over a linear factor)	Substitute: the 0/0 form means factor, cancel the common factor, substitute again.	Practice 5
Factor both the top and the bottom	Factor both completely, cancel the common factor, substitute.	Practice 1
The sign trap a − x = −(x − a)	Rewrite $a - x$ as $-(x - a)$ before cancelling.	Practice 6
Sums and differences of cubes	Use $a^3 \mp b^3 = (a \mp b)(a^2 \pm ab + b^2)$, then cancel.	E02 · Practice 3
After cancelling, the hole is still there	The limit comes from the simplified form; $f(a)$ stays undefined.	E01 · Practice 4, 7
Writing it the AP way (the 0/0 form, lim on every line)	Say “substitution gives the 0/0 form”; keep lim until you substitute.	
Find the constant that makes the limit a real number	The bottom approaches 0, so the top must too: solve top$(a) = 0$.	Practice 2

Example 01

Video · E01

The limit $\lim_{x \to 2} \dfrac{x^2 + ax - 6}{x - 2}$ exists.
Find a and the value of the limit.

Plan: bottom → 0, limit exists → top → 0 → a

Example 02

Video · E02

Let $g(x) = \dfrac{x^3 - 8}{x^2 - 4}$.
Find the limit of $g(x)$ as $x \to 2$. Show the work that leads to your answer.

> Find the top's limit and the bottom's limit first.

VIDEO COMPLETE Check your example answers at the end of this booklet.

1.6a / Example answers

Limits by Factoring

E01 (a) 1; (b) 5

E02 3