1.5b / One-Sided Limits of Piecewise Functions

Watch the lesson ➤

Determining limits using algebraic properties of limits · Part 2

Learning goal

Find one-sided and two-sided limits of piecewise functions at a boundary and away from it, and tell the limit from the value at the boundary.

Key ideas

01 Which formula at a boundary? Take $p(x) = x + 2$ for $x < 2$ and $p(x) = 7 - x$ for $x \geq 2$. $x \to 2$ means $x \neq 2$. From the left, $x < 2$: use $x + 2$. From the right, $x > 2$: use $7 - x$. Each side uses its own **piece**.

02 The value at $x = 2$ itself: find the $=$ sign. $x \geq 2$ has it, so $p(2) = 7 - 2 = 5$. The **value** comes from the piece with $=$.

03 A limit never uses $x = 2$, so $p(2)$ does not count. The limit exists only when both sides head to the same real number. Left $2 + 2 = 4$, right $7 - 2 = 5$, and $4 \neq 5$: no limit. AP: "$\lim_{x \to 2} p(x)$ does not exist, because $\lim_{x \to 2^-} p(x) = 4 \neq 5 = \lim_{x \to 2^+} p(x)$."

04 Near $x = 4$, $x > 2$ on both sides, so only $7 - x$ holds: $\lim_{x \to 4} p(x) = 7 - 4 = 3$. Two sides to compare only at a **boundary**.

05 Absolute value: the bars set the boundary, where the inside is 0. $g(x) = \dfrac{|x - 1| + |x + 1| - 2}{x - 1}$: split at 1; near 1, $|x+1| = x+1$. For $x > 1$, $|x-1| = x-1$ and $g(x) = \dfrac{2(x - 1)}{x - 1} = 2$; for $x < 1$, $|x - 1| = -(x - 1)$ and $g(x) = \dfrac{0}{x - 1} = 0$. Left 0, right 2: $\lim_{x \to 1} g(x)$ does not exist. In general, split at inside $= 0$.

06 A function inside a function, $f(g(x))$ as $x \to 3$: the inside first. $g(x) \to 1^+$ (from above), so use the **right** piece of f near 1, not $f(1)$: $f(g(x)) \to 2$. Ask where the inside goes, and from which side.

Remember

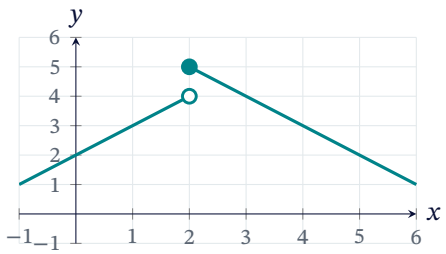
piecewise function
A function given by different formulas on different intervals.

boundary
An x-value where the formula changes.

piece
One formula of a piecewise function, together with the interval where it holds.

absolute value as two pieces
|x − c| equals x − c when x is c or more, and −(x − c) when x is less than c.

Each side has its own piece



For $p(x) = x + 2$ $(x < 2)$ and $7 - x$ $(x \geq 2)$: the left side heads to 4, the right side to 5, and the filled end gives $p(2) = 5$. The limit at 2 does not exist. Near 4, only $7 - x$ holds: $\lim_{x \to 4} p(x) = 3$.

Class notes

1.5b / Write it clearly

What earns the point on the AP Exam, and the mistakes that lose it.

Writing checklist

Notation to use and to avoid

WRITE	AVOID
Name the piece: "For $x < 2$, $f(x) = x^2 + 1$, so $\lim_{x \to 2^-} f(x) = 2^2 + 1 = 5$."	Do not write $\lim_{x \to 2} f(x) = 5$ and 4. A limit is one real number.
Compare the sides in words: "$\lim_{x \to 2^-} f(x) = 5$ and $\lim_{x \to 2^+} f(x) = 4$ are not equal, so $\lim_{x \to 2} f(x)$ does not exist."	Do not write $\lim_{x \to 2^-} f(x) = f(2)$ unless the left piece gives that number.

Common mistakes

The mistake code is used in the answers and in My Mistakes

01 The equals sign gives the value, not the limit

Code L2

✗ For $f(x) = x^2 + 3$ $(x < 1)$ and $2x$ $(x \geq 1)$: $\lim_{x \to 1^-} f(x) = f(1) = 2$.

✓ The left side uses $x^2 + 3$: $\lim_{x \to 1^-} f(x) = 1 + 3 = 4$. The value $f(1) = 2$ comes from the other piece.

02 Use the piece for that side

Code L3

✗ For $g(x) = 5 - x$ $(x \leq 2)$ and x^2 $(x > 2)$: $\lim_{x \to 2^+} g(x) = 5 - 2 = 3$.

✓ $x \to 2^+$ means $x > 2$, so use x^2: $\lim_{x \to 2^+} g(x) = 4$.

03 Absolute value changes sign on the left

Code L4

✗ $\dfrac{|x-5|}{x-5} = 1$, so $\lim_{x \to 5^-} \dfrac{|x-5|}{x-5} = 1$.

✓ For $x < 5$, $|x-5| = -(x-5)$, so the quotient is -1 and the left-hand limit is -1.

Real AP questions on this topic Free on AP Central. Try them after the practice.

No released FRQ asks only for a one-sided limit of a piecewise function. Piecewise functions return in Topics 1.11–1.13.

1.5b / Guided examples

Watch, pause, and write the reasoning. Full solutions are in the video.

Problem types

Read the problem, name the type, then use its method

Type	Method	Problems
A one-sided limit at a boundary of a piecewise function	Pick the piece for that side, then substitute $x = c$.	
Whether the two-sided limit exists at a boundary	Find both one-sided limits; if both equal the same real number L, the limit is L.	Practice 5
The value at the boundary (which piece has the equals sign)	Use the piece whose condition includes c; it does not decide the limits.	E01 · Practice 1
Pieces with exponential, trigonometric, or root formulas	Same steps; know $e^0 = 1$, $2^0 = 1$, $\cos 0 = 1$, $\cos \pi = -1$.	
A limit away from the boundaries	Find the one piece that holds near c; substitute when it is a polynomial, e^x, $\sin x$, $\cos x$, or a root defined there.	
Find k so that the limit exists	Set the left-hand limit equal to the right-hand limit and solve.	E03 · Practice 3
Absolute value written as two pieces	Rewrite $\lvert x - c \rvert$ as $x - c$ for $x > c$ and $-(x - c)$ for $x < c$.	E02 · Practice 2
A sum of two piecewise functions at a boundary	Find each one-sided limit, add side by side (left with left, right with right), then compare the sides.	Practice 4, 6
Track the side of the outer input in a composition	Follow the inside output first. Track which side of the outer input it approaches.	E04 · Practice 7

Example 01

Video · E01

Let $f(x) = \begin{cases} x^2 - 2x, & x < 3 \\ 6, & x = 3 \\ 2x - 3, & x > 3 \end{cases}$

Find (a) $\lim_{x \to 3^-} f(x)$ (b) $\lim_{x \to 3^+} f(x)$
(c) $\lim_{x \to 3} f(x)$ (d) $f(3)$

> Each side uses its own piece. The value comes from the line with $x = 3$.

Example 02

Video · E02

Let $g(x) = \dfrac{(x+1)\,|x-2|}{x-2}$.
Find (a) $\lim_{x\to 2^-} g(x)$ (b) $\lim_{x\to 2^+} g(x)$
(c) $\lim_{x\to 2} g(x)$, or explain why it does not exist.

> The inside is 0 at $x = 2$. Split the bars there.

Example 03

Video · E03

$f(x) = kx - 1\ (x < 2),\ x^2 + k\ (x \geq 2)$
For what k does $\lim_{x\to 2} f(x)$ exist?

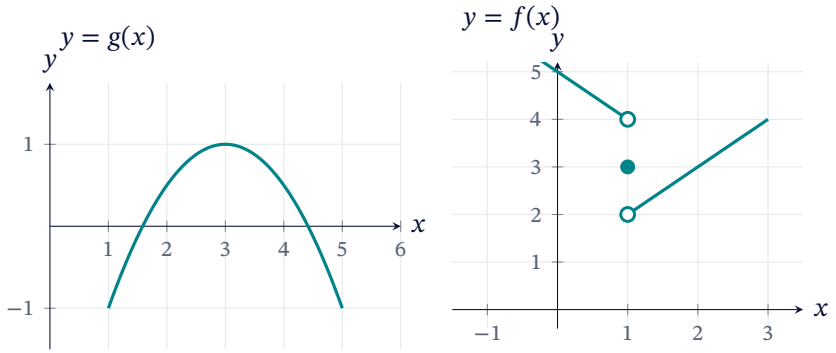
A. 3
B. 4
C. 5
D. no such value

> Write each one-sided limit in terms of k, then set them equal.

Example 04

Video · E04

The graphs of g (top) and f (bottom) are shown. What is $\lim_{x\to 3} f(g(x))$?



A. 1

B. 2

C. 4

D. 3

> Find where the inside graph heads and check which side of that input it stays on.

VIDEO COMPLETE Check your example answers at the end of this booklet.

1.5b / Example answers

One-Sided Limits of Piecewise Functions

E01 (a) 3; (b) 3; (c) 3; (d) 6

E02 (a) -3; (b) 3; (c) does not exist: the left side is -3 and the right side is 3

E03 (C)

E04 (C)