1.5a / Limit Laws

Watch the lesson ➤

Determining limits using algebraic properties of limits · Part 1

Learning goal
Use the limit laws to find limits of sums, differences, products, quotients, powers, and roots, and tell when a law cannot be used.

Key ideas

01 Constant times f: $f(x)$ near 4, so $3f(x)$ near $3 \cdot 4$: $\lim_{x\to 3} 3f(x) = 3(4) = \mathbf{12}$. In general, $\lim_{x\to a} c\,f(x) = cL$.

02 Sum and difference: $\lim_{x\to 3}[f(x) - g(x)] = 4 - (-2) = 6$. In general, $\lim_{x\to a}[f(x) \pm g(x)] = L \pm M$.

03 Product: $\lim_{x\to 3} f(x)g(x) = 4(-2) = -8$. In general, $\lim_{x\to a} f(x)g(x) = LM$ (the product law).

04 Quotient: $\lim_{x\to 3} \dfrac{f(x)}{g(x)} = \dfrac{4}{-2} = -2$. In general, $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, only if $M \neq 0$. If $M = 0$, no **quotient** law: as in $\dfrac{f(x)}{x-3}$, the top heads to 4 and the bottom to 0. Think: $\dfrac{4}{0^+} = \infty$, $\dfrac{4}{0^-} = -\infty$: unbounded, no real-number limit. If the top heads to 0 too, $\dfrac{0}{0}$: not done yet.

05 Powers and roots: $[g(x)]^3 = g(x) \cdot g(x) \cdot g(x)$, so by the product law $\lim_{x\to 3}[g(x)]^3 = (-2)^3 = -8$; in general $\lim_{x\to a}[f(x)]^n = L^n$. $\lim_{x\to 3}\sqrt{f(x)} = \sqrt{4} = \mathbf{2}$; in general $\lim_{x\to a}\sqrt{f(x)} = \sqrt{L}$ for $L > 0$ (the root of a negative number is not real). Also $\lim_{x\to a} e^{g(x)} = e^M$.

06 A polynomial, and a quotient of polynomials whose bottom limit is not 0: plug in.

07 ? Each limit must **exist** as a **real** number, and one-sided limits work the same way, one side at a time.

Remember

limit law
A rule that builds the limit of a combination from the limits of its parts.

quotient law
The limit of f/g is the limit of f divided by the limit of g, when the bottom limit is not 0.

product law
The limit of f times g is the limit of f times the limit of g, when both limits exist.

plug in
Put x = a into a polynomial, or a quotient of polynomials whose bottom is not 0 at a; the limit laws say this gives the limit.

The limit laws (L and M are real numbers)

$$\lim[f \pm g] = L \pm M, \qquad \lim c\,f = cL$$
$$\lim fg = LM, \qquad \lim \frac{f}{g} = \frac{L}{M}\ (M \neq 0)$$
$$\lim f^{\,n} = L^n\ (n \text{ a positive integer}), \qquad \lim \sqrt{f} = \sqrt{L}\ (L > 0), \qquad \lim e^g = e^M$$

Class notes

1.5a / Write it clearly

What earns the point on the AP Exam, and the mistakes that lose it.

Writing checklist

Notation to use and to avoid

WRITE	AVOID
Keep lim on every line until you substitute: $\lim_{x\to 2}(x^2+1) = 2^2+1 = 5$.	Do not write $\lim_{x\to 2} = 5$, and do not drop lim in the middle of the work.
Give the reason when you use the quotient law: "the bottom limit is 4, not 0."	Do not write a limit as $\frac{0}{0}$, and do not do arithmetic with ∞.

Common mistakes

The mistake code is used in the answers and in My Mistakes

01 Use the limits, not the values

Code L2

✗ $\lim_{x\to 4}[f(x)+g(x)] = f(4)+g(4) = 1+6 = 7$, when $\lim_{x\to 4} f(x) = 3$ and $\lim_{x\to 4} g(x) = -2$.

✓ The sum law adds the limits: $\lim_{x\to 4}[f(x)+g(x)] = 3+(-2) = 1$. The values $f(4)$ and $g(4)$ are never used.

02 The quotient law needs a bottom limit that is not 0

Code L1

✗ $\lim_{x\to 5} f(x) = 0$ and $\lim_{x\to 5} g(x) = 0$, so $\lim_{x\to 5} \dfrac{f(x)}{g(x)} = \dfrac{0}{0} = 1$.

✓ 0/0 is not a number, and the quotient law does not apply. The limit depends on f and g themselves (Lessons 1.6–1.8 show how to find it).

03 Infinity is not a number

Code L8

✗ $\lim_{x\to 0} f(x) = \infty$ and $\lim_{x\to 0} g(x) = \infty$, so $\lim_{x\to 0}[f(x)-g(x)] = \infty-\infty = 0$.

✓ The laws need finite limits. With $f(x) = \frac{1}{x^2}+5$ and $g(x) = \frac{1}{x^2}$, $f(x)-g(x) = 5$ for every $x \neq 0$, so the limit is 5, not 0.

Real AP questions on this topic Free on AP Central. Try them after the practice.

No released question asks for a limit law by name. The CED lists sums, differences, products, quotients, and composite functions (LIM-1.D.2) without examples.

1.5a / Guided examples

Watch, pause, and write the reasoning. Full solutions are in the video.

Problem types

Read the problem, name the type, then use its method

Type	Method	Problems
Polynomials and rational functions by direct substitution (from the limit laws)	Check that the bottom limit is not 0, then substitute $x = a$.	
Combining given limits with the limit laws	Each limit exists, so replace it by its number, one law at a time.	E01 · Practice 1
Limits of sums, products and quotients from the graphs of f and g	Read each limit from its graph; if one does not exist, combine side by side.	E02 · Practice 2
When a law cannot be used (a bottom limit of 0, a limit that does not exist, infinity)	Check the conditions first: a bottom limit of 0, a limit that does not exist, or ∞ stops the laws.	Practice 3, 5
Powers, roots and other composites with a basic outer function	Find the inside limit L; check L is allowed ($L \geq 0$ for a square root on its domain); apply the outer function.	Practice 4
Naming the law for each step, with correct limit notation	Name each law and its condition; keep lim until you substitute.	

Example 01

Video · E01

Suppose $\lim_{x\to 1} f(x) = 4$ and $\lim_{x\to 1}[2f(x) - g(x)] = 5$.
(a) Find $\lim_{x\to 1} g(x)$. Show the work.
(b) Is $\lim_{x\to 1} \dfrac{f(x)}{g(x) - 3}$ a real number? Why?

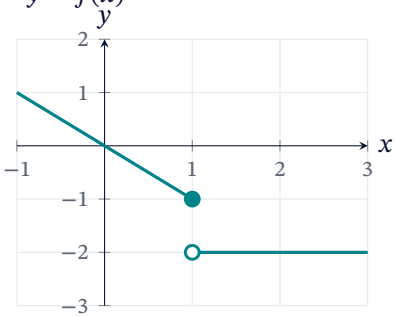
> Write g with pieces whose limits you know.

Example 02

Video · E02

The graphs of f (top) and g (bottom) are shown. What is $\lim_{x \to 1} f(x)g(x)$?

$y = f(x)$



$y = g(x)$



A. 8

B. 4

C. −4

D. nonexistent

> Neither limit exists. Go one side at a time.

VIDEO COMPLETE Check your example answers at the end of this booklet.

1.5a / Example answers

Limit Laws

E01 (a) $\lim_{x \to 1} g(x) = 2(4) - 5 = 3$; (b) No: the numerator approaches 4 and the denominator approaches $3 - 3 = 0$, so the quotient grows without bound in size (think $\frac{4}{0^+}$, $\frac{4}{0^-}$): not a real number

E02 (B)